Text-Book

ON

Differential Calculus

By

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Second Edition

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PREFACE

Advantage has been taken of the opportunity provided by the call for a second edition to add some more explanatory matter, as well as two sets of Miscellaneous Examples, and to base the theory of maxima and minima and of indeterminate forms on Taylor's finite formula instead of the infinite series. A number of minor errors and misprints have also been corrected.

My thanks are due again to most of the persons mentioned in the preface to the first edition for their continued assistance, and to Professor Shanti Narayan, M. A., of Lahore for certain valuable suggestions.

September, 1938 GORAKH PRASAD

PREFACE TO THE FIRST EDITION

This book has been written to meet the requirements of the B. A. and B. Sc. students of Indian Universities. The treatment of the subject is in keeping with the modern theory of functions, but is at the same time simple. The excessive refinements of the present-day higher mathematics have been deliberately excluded, as being unsuitable for the beginner, and geometrical intuition and graphical illustrations have been freely drawn upon to drive home to the student results obtained by analytical methods. With the exception of the last chapter and of a few articles here and there, the whole of the subject-matter here presented has stood the test of actual class-room use for a number of years, and the author believes that the book would be found to be in every way suitable for the B. A. and B. Sc. classes.

The concept of limits, usually found very difficult by students, is derived in Chapter I from the notion of continuity, in a way which, it is believed, will make it very simple. The great variety of applications of the

Differential Calculus to various branches of knowledge, which is given in Chapter IV, immediately after the student has learnt differentiation, will, it is expected, create in him a more lively interest towards the subject than is possible when it is attempted to teach the entire principles of the Calculus first. The subject of Asymptotes has been developed as a natural extension of that of tangents and will be found perhaps easier to grasp than when treated otherwise. An attempt has been made to make the rules regarding Curve Tracing so systematic as to make this subject quite easy. The historical and biographical notes interspersed throughout the book and the short historical sketch at the end will, it is hoped, prove interesting. The historical sketch will serve also to give the student the right outlook as regards 'infinitely small quantities' and related topics.

The book contains just a little more than is necessary for the usual course, and no hesitation need be felt in omitting some of the articles. The number of exercises will also be found to be ample. Of these some are original, a large number has been taken from the examination papers of various universities, and the others are such as are common to practically all text-books on the subject. Some of the exercises have been taken from the excellent collection of Leib, entitled 'Problems in the Calculus.'

I am greatly indebted to Mr. Harish Chandra Gupta, M. SC., a pupil of mine, who has read with great care the proofs of the whole of the book and verified most of the examples. My thanks are due also to my colleagues Dr. B. N. Prasad, PH. D., D.S.C., and Mr. R. N. Choudhuri, B.A. (Cantab.), to my friends Mr. S. D. Seth, M.SC., Mr. R. S. Varma, M.SC., and Mr. R. D. Misra, M.A., to my pupils Mr. Lakshmi Narayan Sharma, M.SC., and Mr. Sadanand Mukerji, M.SC., and to some of the students of my B.Sc. class, who have all very generously helped me in reading the proofs or verifying the examples, or have offered valuable criticism and advice.

University of Allahabad
September, 1938

GORAKH PRASAD

CONTENTS

DIFFERENTIAL CALCULUS

CHAPTER I

LIMITS

1·1. Related Quantities. The area of a circle depends upon its radius; the velocity of a falling particle depends upon the distance through which it has fallen; the cube of a number depends upon the number itself. In these examples the radius of the circle, the distance through which a particle has fallen, and the number which is cubed are variable. They are not fixed. The quantities which depend upon them—area, velocity and cube respectively—also vary.

Differential Calculus deals with the way in which one quantity varies when the other, on which it depends, is made to vary, and with allied topics. But to get a theory which would be applicable to any pair of related quantities, we take up the study of pure numbers instead of concrete quantities; and to avoid any looseness in the arguments we begin by carefully defining the words we shall use. It should be noticed that the word variable is used in a somewhat more restricted sense than the one attached to it in everyday speech.

1·11. Definitions. *A symbol which can take every numerical value, or every numerical value from one given number to another, is called a* **variable.**

If we desire to consider only those numerical values of x which lie between the two given numbers a and b, then all the numerical values between a and b taken collectively will be called the **domain** of the variable. The numbers a and b themselves are also generally included in the domain. The domain is usually denoted by (a, b). We say that the domain of x is (a, b). Very often, however, we want to consider every numerical value, and not only those which lie between two given numbers. In such a case the domain is generally not mentioned.

A symbol which retains the same value throughout a set of mathematical operations is called a **constant.**

Let x be a variable. Then *a symbol which has one definite value for every value of x is called a* **function of x.**

It is customary to denote constants by the earlier letters of the alphabet, $a, b, c, \ldots$. The letters $x, y, z, u, v, \ldots$ are generally used for variables.

A function of x may be, and very often is, denoted by a single letter, such as y. More generally it is denoted by symbols like $f(x)$, or $F(x)$, or $\varphi(x)$. The value of the function $f(x)$ for $x = a$ is denoted by $f(a)$. A particular function is actually defined by giving a rule, or a set of rules, by which the value of y can be computed for every value of x. It is not necessary that there should be only one formula for the whole domain of x.

Ex. 1. If y is always equal to x^2, then y is a function of x, for the above definition is evidently satisfied. Similarly $\cos x$, e^x, $\log x$, $(x-a)^n$, etc., are functions of x.

Ex. 2. If y be defined by saying that $y = x^2$ when x is greater than 2, and $y = x - 1$ when x is not greater than 2, then y, thus defined, is a function of x; for y has a definite value for every value of x. Here two different formulæ have been used to define one function of x, of which one formula holds for one part of the domain, and another holds for the remaining part of the domain.

Ex. 3. The number of students in a class is not a "variable" of the type contemplated above, because it cannot take up a fractional value like 30·2.

Ex. 4. "$y = x^3$ when x is integral" is not enough to make y a function of x in the sense contemplated above, because we do not know the value of y when x is not integral.

1·12. Remarks. (1) *If y is a function of x, y is sometimes called the* **dependent variable** *and x the* **independent variable.** This nomenclature is based on the fact that the value of x can be arbitrarily chosen. Then y has a value which depends upon the chosen value of x. The word variable in "the dependent variable" is not used in the exact sense implied in the definition of a variable given above. This is apparent from Example 2 of the previous article, in which y cannot take up the values between 1 and 4.

(2) In the definition of a "variable", only real values are meant. A real value might be 0, or some positive or negative number, integral or fractional, rational or irrational.

(3) In the definition of a function y is required to have a definite value. Values like 0/0, 5/0, a/0 are inadmissible. These have

no meaning. Some students are under the impression that $5/0$ is "infinity." This is wrong. Infinity is not a number, and has no meaning except in the sense defined later in Art. 1·4. Nor is $0/0$ equal to 1. If we suppose $0/0$ to be equal to 1, we shall be led to absurdities. Consider, for example, what would be the result of dividing by zero both sides of an equation like $5 \times 0 = 2 \times 0$.

Again an expression like

$$\frac{x^2 - 1}{x - 1}$$

fails to give a definite value when $x = 1$. The numerator and the denominator both become zero when $x = 1$. The student must not think that we can get the value of $(x^2 - 1)/(x - 1)$ by first dividing the numerator by the denominator, and then putting $x = 1$; for *we can divide by $x - 1$ only when x is not equal to 1.*

(4) If $y = x^2$ when x is greater than 2 and $y = x$ when x is less than 2, is y a function of x? Here it is not stated what the value of y is when $x = 2$. Hence the definition of a function is not satisfied, which requires that y should have a definite value for *every* value of x. But for the sake of convenience we still call y a function of x, but say that *y is not defined for $x = 2$.*

(5) If the expression or equation which defines y gives more than one value of y for each value of x, we shall regard the expression or equation as defining more than one function of x.

Thus, if $|a|$ be used to denote the positive value of a, so that $|+4| = +4$ and $|-4| = +4$, the square root of x might be any one of the two functions $|\sqrt{x}|$ and $-|\sqrt{x}|$. In such a case it is understood that the same function is to be taken throughout a particular discussion.

(6) What we have called a variable is generally called a *continuous variable*. Again, if (a, b) is the domain of x, the domain is called a *closed domain* if the numbers a and b also belong to the domain. If a and b do not belong to the domain, then the domain (a, b) is called an *open domain* Also, when y has more than one value for every value of x, then y is generally called a *multiple-valued function* of x.

For the sake of convenience we have adopted somewhat simplified definitions.

1·13. Graphical Representation. Let y be a function of x. Choose a particular value x_1 of x. Let the corresponding value of y be y_1. Take now rectangular axes OX and OY as usual and plot the point (x_1, y_1). By plotting such points for all possible values of x we get a graph of the function. (In practice, points corresponding to values of x at very short intervals, connected by a smooth curve, suffice.) A graph is a great help in

the study of functions, but it cannot always be drawn, and at its best it is a crude method of exhibiting the functional relation between x and y. We shall give, therefore, an entirely arithmetical treatment, and use graphs only to *illustrate* the various definitions and processes.

Ex. 1. The graph of the function $y = x^2$ for values of x from -2 to 2 is shown in Fig. 1. We know from Coordinate Geometry that it is a parabola.

Ex. 2 If $y = x^2$ when x is not equal to 1 and $y = 2$ when $x = 1$, the graph is different from that of the previous example only in the omission of the point (1, 1) from the parabola. Instead of this point we have now the isolated point (1, 2). See Fig. 2.

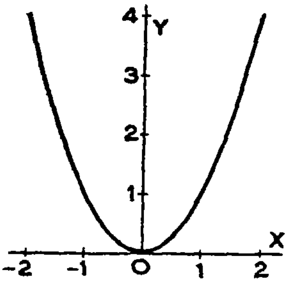


Fig. 1

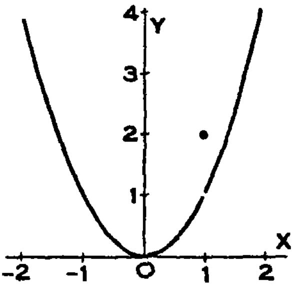


Fig. 2

Ex. 3. If $y = x^2$ when x is less than 1, and $y = x^2 - \frac{1}{2}$ when x is not less than 1, the graph is as shown in Fig. 3, as the student can easily verify.

Examples

1. In which of the following cases is x a continuous variable, i.e., a variable satisfying the definition of § 1·11?

(i) $x =$ time elapsed since a given instant, (ii) $x =$ the distance of a point on a straight line from a given point on the straight line, (iii) x can take all integral values from -10 to $+100$, (iv) x can take all numerical values, between 0 and 1, which can be represented by a terminating decimal.

2. For what value or values of x, if any, are the following functions not defined?

(i) x^3 (ii) $1/x^3$ (iii) $\cos x$ (iv) $\tan x$ (v) $(\sin x)/x$ (vi) $(x^2 - 2x + 1)/(x^2 - 3x + 2)$.

3. Draw a graph of the function y if $y = 0$ when x is an integer and $y = x$ when x is not an integer.

[The graph is derived from the straight line $y = x$ by taking out the points

$$\ldots, (-1, -1), (0, 0), (1, 1), (2, 2), \ldots$$

and adding the points

$$\ldots, (-1, 0), (0, 0), (1, 0), (2, 0), \ldots$$

on the axis of x.

The reader may possibly regard this as an unreasonable function. *Why*, he may ask, if y is equal to x for all values of x save integral values, should it not be equal to x for integral values too? The answer is simply, *why should it?* The function y does in point of fact answer to the definition of a function; there is a relation between x and y such that when x is known y is known. We are perfectly at liberty to take this relation to be what we please, however arbitrary and apparently futile. The function y is, of course, a quite different function from that one which is always equal to x, whatever value, integral or otherwise, x may have.—Hardy: *A Course of Pure Mathematics*.]

4. Draw a graph of the function y from $x = -1$ to $x = 4$ if $y = \mathrm{I}(x)$, where $\mathrm{I}(x)$ denotes the integral part of x.

5. Give two examples of functions of x in which we have to restrict the domain of x in order to keep the function real.

6. If $y = 5$ for every value of x, can y be regarded as a function of x?

1·2. Discontinuity. We see that the curve in Fig. 1 is continuous, i.e., there is no break in it. The curve in Fig. 2 is broken at the point for which $x = 1$. We say there is a *discontinuity* at $x = 1$. In Fig. 3 also there is a discontinuity at $x = 1$.

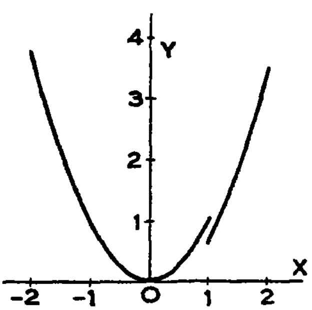


Fig. 3

1·21. A criterion of continuity. How shall we define continuity arithmetically, so that drawing a graph may not be necessary? We notice that for the first function ($y = x^2$ for all values of x) the value of y for $x = 1$ differs only slightly from the neighbouring values of y. Thus, if $OM = 1$ (Fig. 4), the ordinates PM and $P'M'$ differ only slightly if MM' is small. Not only this: *we can make the difference between PM and P'M' as small as we like by making MM' sufficiently small; and, what is* [illegible] *important, the difference between PM and the ordinates at di*[illegible]

points between M and M' will be at least equally small. (This remains true also when we take M' on the other side of M, as in Fig. 5.) Can we say the same thing of the second function, which is discontinuous at $x = 1$ ($y = x^2$ when $x \neq 1$; $y = 2$ when $x = 1$, see Fig. 6)?

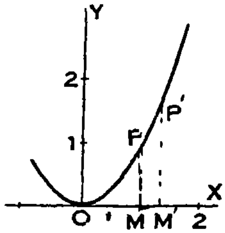


Fig. 4

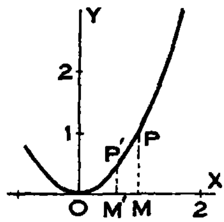


Fig. 5

Here PM differs by about 1 from the neighbouring values of y. We cannot make the difference between PM and $P'M'$ small by making MM' small, whether we take M' on one side of M (Fig. 6) or on the other (Fig. 7).

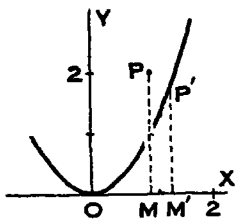


Fig. 6

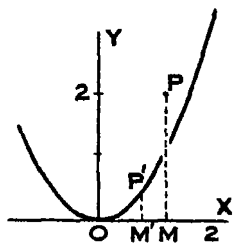


Fig. 7

Here then there is a criterion which we can utilise for distinguishing continuity.

1·22. Meaning of $|x - a| < \delta$. Before we proceed further, however, with the arithmetical treatment of continuity, it is necessary to examine the meaning of the inequality $|x - a| < \delta$.

Now $|x - a|$ means the numerical (i.e., the positive) value of $x - a$, without regard to sign. So $|x - a| < \delta$ means that the difference between x and a, taken positively

is less than δ where, of course, δ must be a positive number. This statement can be broken into two parts :—

(i) if $x > a$, then $x - a < \delta$,
and (ii) if $x < a$, then $a - x < \delta$.

By transposition we see that these are equivalent to the following :—

If $x > a$, then $x < a + \delta$, and if $x < a$, then $x > a - \delta$, i.e., whether x is greater than a, or less than a, x must lie between $a - \delta$ and $a + \delta$.

Hence $|x - a| < \delta$ *means that* x *can have any value between* $a - \delta$ *and* $a + \delta$.

If x be represented in the usual manner by that point on the straight line OX which is at the distance x from the origin O, and if $OA = a$, and $P'A = AP = \delta$, then $|x - a| < \delta$ would simply mean that the point representing x can lie anywhere between P and P'. (The points P and P' are excluded.)

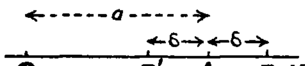


Fig. 8

1·23. Continuity. Arithmetical Definition.

We can now put the criterion of continuity (of § 1·21) in an arithmetical form.

Let $OM = a$ (see Fig. 6) and $OM' = x$. Let $P'M' = f(x)$, so that $PM = f(a)$. The difference between PM and $P'M'$, taken positively is $|f(x) - f(a)|$. We want that it should be possible to make this as small as we please. In other words, it should be possible to make $|f(x) - f(a)| < \epsilon$, where ϵ is an arbitrarily chosen small positive number.

Now we should be able to make $|f(x) - f(a)| < \epsilon$ by choosing MM' to be sufficiently small, and M' might be on one side of M or on the other. We can express the same thing in symbols by saying that we should be able to make $|f(x) - f(a)| < \epsilon$ by finding a sufficiently small positive number δ and restricting x to those values only for which

$$|x - a| < \delta.$$

If it is possible to find such a δ, the function would be continuous; if it is impossible to find any such δ, the function must be discontinuous.

We are thus led to the following definition of continuity :—

A function $f(x)$ of the variable x is said to be **continuous** *at $x = a$ if, for any arbitrarily chosen positive number* ϵ, *however*

small (but not zero), we can find a corresponding number δ *such that*

$$|f(x) - f(a)| < \varepsilon,$$

for all values of x *for which*

$$|x - a| < \delta.$$

The value of δ would, of course, depend on that of ϵ.

Ex. Show that $\sin x$ is continuous for every value of x.

Let a be any value of x between 0 and $\frac{1}{2}\pi$ (both inclusive). $\sin x$ will be continuous at $x = a$ if we can find a δ such that

$$|\sin x - \sin a| < \epsilon$$

for all values of x for which

$$|x - a| < \delta.$$

I. Suppose $x > a$. Let x be equal to $a + \alpha$, where α is positive and small.

We must have $\quad \sin x - \sin a < \epsilon,$

or $\quad \sin(a + \alpha) - \sin a < \epsilon,$

or $\quad \sin a \cos \alpha - \sin a + \cos a \sin \alpha < \epsilon,$

or $\quad -\sin a . \, 2 \sin^2 \frac{1}{2}\alpha + \cos a \sin \alpha < \epsilon.$

It will be sufficient if we make

$$+ \sin a . \, 2 \sin^2 \tfrac{1}{2}\alpha + \cos a \sin \alpha < \epsilon;$$

and, because $\sin a$ and $\cos a$ do not exceed 1, it will be quite sufficient if we make

$$2 \sin^2 \tfrac{1}{2}\alpha + \sin \alpha < \epsilon.$$

As $\sin \frac{1}{2}\alpha < 1$, so $\sin^2 \frac{1}{2}\alpha < \sin \frac{1}{2}\alpha$. Hence it will be amply sufficient if we make

$$2 \sin \tfrac{1}{2}\alpha + \sin \alpha < \epsilon.$$

Now α may be supposed to lie between 0 and $\frac{1}{2}\pi$ since α is small. Therefore, $\sin \alpha < \alpha$. Hence, it will be sufficient to make

$$2(\tfrac{1}{2}\alpha) + \alpha < \epsilon,$$

or $$\alpha < \tfrac{1}{2}\epsilon.$$

We see, therefore, that if we take δ to be equal to $\frac{1}{2}\epsilon$, i.e., if

$$x - a < \tfrac{1}{2}\epsilon,$$

then $$\sin x - \sin a < \epsilon.$$

II. Next suppose $x < a$. Let $x = a - \alpha$, where α is positive (and small). We can show as above that

$$\sin a - \sin(a - \alpha) < \epsilon$$

if $$a - x < \tfrac{1}{2}\epsilon.$$

III. Finally suppose $x = a$. Then evidently

$$| \sin x - \sin a | < \epsilon,$$

because the left hand side is zero and the right hand side is positive.

We have thus shown that if

$$| x - a | < \tfrac{1}{2}\epsilon,$$

then $$| \sin x - \sin a | < \epsilon.$$

This is true however small ϵ may be. Hence $\sin x$ is continuous at $x = a$.

If a does not lie between 0 and $\frac{1}{2}\pi$, we can use the periodic properties of $\sin x$ to show that $\sin x$ is continuous at $x = a$.

Hence $\sin x$ is continuous for every value of x.

1·24. The value of a function for $x = a$ is independent of its values in the neighbourhood. Suppose a function is defined as follows:—

$y = x^2$ when x is less than 1; $y = x^2$ when x is greater than 1.

This function is not defined for $x = 1$. The question arises if we can find its value when $x = 1$. The answer must be in the negative, for this function might be the same as the one considered in Example 1 of § 1·13, or it might be the same as the one considered in Example 2 of that article, or it might be still another function. In the first case $y = 1$ when $x = 1$; in the second $y = 2$ when $x = 1$. These are not the only possibilities, for y might be $\frac{1}{2}$, or -3, or 1 million, or in fact any other number, when $x = 1$. We see that *the value of a function for $x = a$ (where a is any number), does not depend upon the value of the function for neighbouring values of x.* It depends entirely upon the definition of the function and so cannot be found if the function is undefined for $x = a$.

1·3. Limit. Consider the function $y = x + 1$. Its graph is a straight line.

When $x = 1$, the value of y is 2.

Let us now for a moment disregard what the value of the function is for $x = 1$, and consider the values of the function for only the other values of x. As a help in fixing ideas, we may draw the graph, and consider all points on the straight line $y = x + 1$, except the point $(1, 2)$. See Fig. 9.

Consider now the question: what value of y for $x = 1$ will make the graph continuous at $x = 1$? The answer evidently is 2; i.e., 2 is that number the difference between which and y can be made less than an arbitrarily chosen small positive number ϵ by taking $|x - 1|$ sufficiently small.

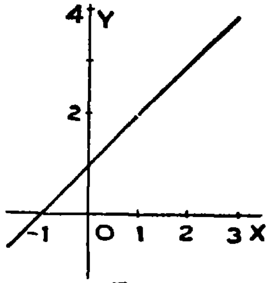


Fig. 9

This number 2 is called the *limit* of y at $x = 1$.

In this case the *value* of y at $x = 1$ is also 2. This is mere coincidence. The value of y at $x = 1$ does not depend on the neighbouring values of y (§ 1·24), whilst the limit of y at $x = 1$ depends solely on the neighbouring values of y; so these would not always be the same.

Consider, for instance, Ex. 2 of § 1·13 ($y = x^2$ when $x \neq 1$; $y = 2$ when $x = 1$). By making $|x - 1|$ sufficiently small we can make $|y - 1|$ less than ε, but not $|y - 2|$; so the *limit* of y at $x = 1$ is 1. But the *value* of y at $x = 1$ is 2. The limit and the value are here different.

Take a third example. Let

$$y = \frac{x^2 - 1}{x - 1}.$$

Here y is not defined at $x = 1$ (§ 1·12). But if $x \neq 1$, we can divide out by $x - 1$, getting $y = x + 1$. The graph must therefore be as shown in Fig. 9. Hence the limit of y at $x = 1$ is 2, as in the first example. Thus the limit of y at $x = 1$ can be found even when y is not defined for $x = 1$.

Keeping the definition of continuity in mind, we can, therefore, define a limit as follows:—

If $f(x)$ is a function of the variable x, the number A is said to be the limit of $f(x)$ for $x = a$ if, for any arbitrarily chosen positive number ε, however small but not zero, there exists a corresponding number δ greater than zero such that

$$|f(x) - A| < \varepsilon,$$

for all values of x for which

$$0 < |x - a| < \delta.$$

We have added here the condition $0 < |x - a|$, i.e., we have excluded the possibility of x becoming equal to a itself. This is necessary, because if the function is discontinuous, as in Fig. 6 (p. 6), then the A which makes $|f(x) - A|$ less than ϵ when x is in the neighbourhood of a would differ by a definite amount from $f(a)$. Thus in Fig. 6, A as deduced from the values in the neighbourhood of $x = 1$ is 1, but $f(1)$ itself is 2, and we cannot make $|f(1) - 1|$ less than ϵ, where ϵ is arbitrarily small.

1·31. Geometrical Language. The representation of functions by graphs is so well established that we very often employ geometrical language even for mere numbers. Thus we very often say "at the point $x = a$," when we mean "for the value a of x." Similarly, if A is the limit of $f(x)$ for $x = a$, it is not uncommon to express the same thing by saying that "*$f(x)$ approaches A as x approaches a*". It is clear from the accompanying figure that this sentence expresses the same geometrical idea as the one we had considered before, viz., that the difference between $P'M'$ (which is equal to $f(x)$) and QM (which is equal to A) can be made as small as we please by making MM' sufficiently small (whether M' be on one side of M or on the other) and the difference between QM and the ordinate at *any* point between M and M' will also be at least equally small.

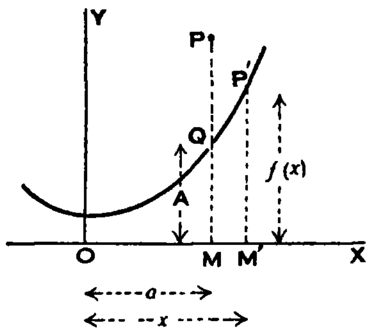


Fig. 10

Much oftener we use the phrase "tends to" instead of the word "approaches," the idea of motion being behind this phrase too. We say "*$f(x)$ tends to A as x tends to a*" if the limit of $f(x)$ for $x = a$ is A; or say that "*the limit of $f(x)$, as x tends to a, is A.*" We write

$$\lim_{x \to a} f(x) = A, \text{ or } \lim_{x \to a} f(x) = A.$$

The arrow can be read as "tends to." It has already been made clear that $\lim_{x \to a} f(x)$ does not depend on the value of $f(x)$ for $x = a$. From this point of view, the notation $\lim_{x \to a}$ is good, as

the arrow constantly reminds the beginner that we are not concerned with the value when $x = a$. The older books, however, use the notation:

$$\lim_{x=a} f(x) = A,$$

which is rather misleading. Moreover, they very often use "becomes" or "ultimately becomes" for "tends to," which is equally misleading. The student begins to think that $\lim_{x=a} f(x)$ must be the same as the value of $f(x)$ when $x = a$, and in this erroneous idea he is aided by the fact that usually $\lim_{x \to a} f(x)$ is equal to $f(a)$ as in Fig. 4 (p. 6).

1·33. A limit does not necessarily exist. It is not necessary that the limit of $f(x)$ should exist when x approaches a given value. A simple example will make this clear.

Consider the function defined as follows:—

$$\begin{aligned} y &= x^2 && \text{when } x < 1, \\ y &= 1{\cdot}5 && \text{when } x = 1, \\ y &= x^2 + 1 && \text{when } x > 1. \end{aligned}$$

These three equations define only one function of x (§ 1·11), whose graph evidently is as shown in Fig. 11 or 12. If x approaches 1 from the right, as in Fig. 11, $f(x)$, i.e. $P'M'$, evidently approaches the value $1^2 + 1$, i.e., 2.

But if x approaches 1 from the left, as in Fig. 12, $f(x)$ evidently approaches the value 1.

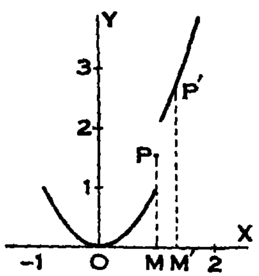


Fig. 11

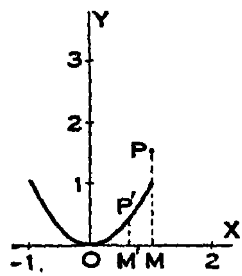


Fig. 12

Thus there is no single number to which $f(x)$ tends, irrespective of the consideration whether x tends to 1 from the right or from the left. Or, in other words, there is

no number A such that $|f(x) - A| <$ an arbitrarily small number ε, whenever $0 < |x - 1| < \delta$. Hence we say that $\lim_{x\to 1} f(x)$ does not exist in this case.

Of course, we can find a number A_R say, such that $|f(x) - A_R| < \epsilon$ when $0 < x - 1 < \delta$, and we can find another number, A_L say, such that $|f(x) - A_L| < \epsilon$ when $0 < 1 - x < \delta$. The numbers A_R and A_L would be called the limit of $f(x)$ at 1 on the right and the limit of $f(x)$ at 1 on the left respectively. But these limits are not very important for an elementary text-book like the present one.

1·34. A function for which the limit at x = 0 does not exist. For the above function the limit on the right and the limit on the left both exist. Only they are not equal, and so we have to say that the limit at the point under consideration does not exist.

But there are functions for which, for a certain value or values of x, neither the limit on the right exists, nor the limit on the left exists.

Consider, for example, the function defined by

$$y = \sin \frac{1}{x}.$$

To draw the graph, we notice that when $x = 1$, $y = \sin 1$, i.e., $y = \sin$ (1 radian) = about 0·84. As x takes up smaller values, $1/x$ takes up all the values from 1 onwards and thus passes through the values $\frac{1}{2}\pi$ (= about 1·57), π (= about 3·14), $\frac{3}{2}\pi$, 2π, etc., and as x gets still smaller, $1/x$ gets still larger and passes through such

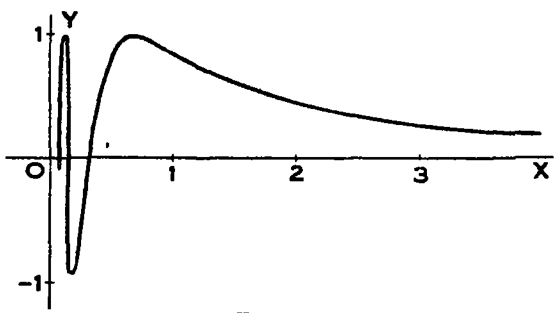


Fig 13

values as 1 billion π, 1 billion $\pi + \frac{1}{2}\pi$, etc. (Of course, when $x = 0$, $1/x$ has no meaning, and the function is not defined at $x = 0$; but we are not concerned with this at present). It follows that as

x tends to zero, after starting from 1, the value of y oscillates between $+1$ and -1, passing through 0 and the intermediate values, an indefinitely large number of times. We can never hope to draw the complete graph up to $x = 0$, but a portion is shown in the figure.

Evidently there is no number A to which $f(x)$ tends as x tends to zero from the right. So the limit on the right does not exist. Similarly, the limit on the left also does not exist.

Hence the ordinary limit of $\sin(1/x)$, without any qualification as to right or left, also does not exist at $x = 0$.

1·4. Infinite Limits. Consider the function y which is equal to $1/x^2$ for every value of x.

When x is very small, y is very large. If we take smaller and smaller values of x, y gets larger and larger. As x approaches 0 (either from the right or the left), there is no barrier to the increase of $1/x^2$. We cannot specify a number N and say that $1/x^2$ will remain less than N; for, however large N may be, we can make $1/x^2$ greater than N by choosing x small enough. We say, therefore, that $1/x^2$ *tends to infinity (or the limit of* $1/x^2$ *is infinity) as* x *tends to zero.*

Similarly, $1/(x-a)^2$ tends to infinity as x tends to a. The symbol for infinity is ∞.

$-1/(x-a)^2$ is said to tend to $-\infty$ (minus infinity) as $x \to a$, because as $x \to a$, $-1/(x-a)^2$ takes up numerically larger and larger, but negative, values.

The discussion given in Articles 1·22, et seq., enables us to put the above definition in precise arithmetical language as follows:—

$\text{Lim}_{x \to a} f(x)$ is said to be infinity, if, for any arbitrarily chosen positive number N, however large, there exists a corresponding number δ greater than zero such that $f(x) > N$ for all values of x for which $0 < |x - a| < \delta$.

1·41. Behaviour of $1/x$ as $x \to 0$. From the preceding article it follows that the limit of $1/x$ as $x \to 0$ from the right is $+\infty$; the limit as $x \to 0$ from the left is $-\infty$.

Hence *the limit of* $1/x$ *as* $x \to 0$ *does not exist.*

In the same way the limit of $1/(x-a)$ as $x \to a$ does not exist.

1·42. Behaviour of $1/x$ as x tends to infinity. Consider the values of the function $1/x$ as we take larger and larger values of x. We know that $1/x$ becomes smaller

and smaller. If we take a sufficiently large value of x, we can make the difference between $1/x$ and zero as small as we please. We express this by saying that *the limit of $1/x$ as x tends to infinity is zero.*

1·43. Limit as $x \to \infty$. If $f(x)$ tends to A as x gets larger and larger in such a way that we cannot assign any number N and say that all the values of x are less than N, we say that the limit of $f(x)$ as x tends to infinity is A.

We can express this more precisely in arithmetical language as follows :—

$\text{Lim}_{x \to \infty} f(x)$ is said to be A, if, for any arbitrarily chosen positive number ϵ, however small but greater than zero, there exists a corresponding positive number N such that

$$|f(x) - A| < \epsilon$$

for all values of x greater than N.

A similar definition can be given for x tending to $-\infty$.

1·44. Misleading nomenclature. The expressions "x tends to infinity" and "x approaches infinity" are likely to mislead a beginner unless he is on his guard. These expressions perhaps suggest that infinity is some number like 1 million or 1 billion, only very much larger; and x tends to infinity in the sense that the difference between it and infinity goes on diminishing. This is wrong. The expression "x *tends to infinity" means merely that x goes on increasing, and there exists no number N than which all these values of x are less.*

"$f(x)$ tends to infinity" means a similar thing.

It should be remembered that even when we say that the number of terms in the series

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \dots$$

is infinite, we merely mean that the number of terms is not finite. In other words there is no last term; there are terms after $1/2^n$ however large n may be.

We never mean that we can count the number of the terms and their total number is equal to the number infinity.

NOTE. "$1/x^2 = \infty$ when $x = 0$" is an erroneous way of saying that $\lim_{x\to 0}(1/x^2) = \infty$.

To say that "$1/x = \infty$ when $x = 0$" is doubly wrong, because $\lim_{x\to 0}(1/x)$ does not exist (§ 1·41).

1·5. Continuity : alternative definition. It should be clear now that we may define continuity also as follows:

A function $f(x)$ is said to be continuous at $x = a$ if $\lim_{x \to a} f(x)$ exists, is finite, and is equal to $f(a)$. Otherwise the function is discontinuous at $x = a$.

Ex. 1. $\sin(1/x)$ is discontinuous at $x = 0$, for

$$\lim_{x \to 0} \sin(1/x)$$

does not exist (§ 1·34).

Ex. 2. $1/x^2$ is discontinuous at $x = 0$, for $\lim_{x \to 0}(1/x^2)$ is not finite (§ 1·4)

Ex. 3. If $y = x^2$ when $x \neq 1$ and $y = 2$ when $x = 1$, the function y thus defined is discontinuous at $x = 1$, for $\lim_{x \to 1} f(x)$ is not equal to $f(1)$.

1·6. Continuous Function. *A function of x which is continuous for every value of x in the domain (a, b) is called a continuous function of x in the domain (a, b).*

If $f(x)$ is continuous at $x = a$, then, according to the definition, $\lim_{x \to a} f(x) = f(a)$. Hence we can find $\lim_{x \to a} f(x)$ merely by substituting a for x in $f(x)$, provided $f(x)$ is continuous at $x = a$.

1·61. Continuity of the elementary functions. The student is supposed to be familiar with the elementary functions and to know that :—

(i) x^n is continuous for all values of x when n is positive.

(ii) x^n is continuous for all values of x except $x = 0$ when n is negative. For, when n is negative and equal to $-m$ say, where m is positive, we can write x^n as $1/x^m$, and as $x \to 0$, $1/x^m$ either does not tend to a limit (cf. § 1·41) or $\to \infty$.

(iii) $\sin x$ and $\cos x$ are continuous for all values of x.

(iv) $\tan x$ is continuous for all values of x except $\frac{1}{2}\pi$, $\frac{3}{2}\pi$, $\frac{5}{2}\pi$, etc. As $x \to \frac{1}{2}\pi$ from the right, $\tan x \to +\infty$. As $x \to \frac{1}{2}\pi$ from the left, $\tan x \to -\infty$. Thus $\lim_{x \to \frac{1}{2}\pi} \tan x$ does not exist.

(v) Similar statements can be made for $\cot x$, $\sec x$ and $\operatorname{cosec} x$.

(vi) The inverse circular functions have an indefinitely large number of values for every value of x, and so some restriction is necessary to make them one-valued. For the sake of convenience, throughout this book we shall suppose, unless there is an express statement to the contrary, that $\sin^{-1}x$, $\tan^{-1}x$, $\cot^{-1}x$, and $\operatorname{cosec}^{-1}x$ lie between $-\frac{1}{2}\pi$ and $\frac{1}{2}\pi$ (both values inclusive); and $\cos^{-1}x$, $\sec^{-1}x$, and $\operatorname{vers}^{-1}x$ lie between 0 and π (both values inclusive).

(vii) $\sin^{-1}x$ and $\cos^{-1}x$ are not defined for values of x outside the domain $(-1, 1)$. $\sec^{-1}x$ and $\operatorname{cosec}^{-1}x$ are not defined for values of x between -1 and 1 (both values exclusive).

(viii) All the inverse circular functions are continuous at every point within the domain for which they are defined.

(ix) $\log x$ is continuous for all values of $x > 0$. For values of $x < 0$, the function is not real and so we shall not consider these values. As x tends to zero from the right, $\log x$ tends to $-\infty$.

1·7. Five theorems about limits. The truth of the following theorems would be taken for granted. They appear to be self-evident to a beginner, and for that very reason the proof would be too subtle for him. If, as $x \to a$, $\varphi(x) \to A$ and $\psi(x) \to B$, then, in general, as $x \to a$,

$$\text{(i)} \qquad \{\varphi(x) + \psi(x)\} \to A + B,$$

$$\text{(ii)} \qquad \{\varphi(x) - \psi(x)\} \to A - B,$$

$$\text{(iii)} \qquad \{\varphi(x) \cdot \psi(x)\} \to A.B,$$

$$\text{(iv)} \qquad \varphi(x)/\psi(x) \to A/B,$$

$$\text{and} \qquad \text{(v)} \qquad \varphi(x)^{\psi(x)} \to A^B.$$

There are certain exceptional cases. Thus, if $\phi(x) \to 0$ and $\psi(x) \to 0$ as $x \to a$, then (iv) would imply that $\phi(x)/\psi(x) \to 0/0$, which has no meaning. These exceptional cases would be considered in a later chapter (Chap. XIV). Moreover, in (v), $\phi(x)$ must be positive. This is because (v) is proved by taking logarithms of both sides. If $\phi(x)$ is negative, $\log \phi(x)$ is not real and difficulties arise, specially in the interpretation of $\phi(x)^{\psi(x)}$.

These propositions can be easily extended to cover cases where more than two functions are involved. Thus, if $\phi_1(x), \phi_2(x), \dots \phi_n(x)$ tend to $A_1, A_2, \dots A_n$ respectively as $x \to a$, then in general, as $x \to a$,

$$\phi_1(x) + \phi_2(x) + \dots + \phi_n(x) \to A_1 + A_2 + \dots + A_n.$$

The student must not suppose that the theorem will necessarily hold even if the number of terms be infinite. Consider for example the sum

$$\frac{x^2}{1 + x^2} + \frac{x^2}{(1 + x^2)^2} + \frac{x^2}{(1 + x^2)^3} + \dots \textit{ad inf.}$$

As the terms are in geometrical progression with the common ratio $1/(1 + x^2)$, which is less than unity if $x \neq 0$, it is easily seen that the sum is 1 if $x \neq 0$. Hence the limit of this sum as x tends to zero, which depends only on the values of this sum for values of x other than zero, is 1. But $x^2/(1 + x^2)$, $x^2/(1 + x^2)^2$, ... all $\to 0$ as $x \to 0$. Hence one would ordinarily expect the limit of the sum as $x \to 0$ to be 0.

1·71. Continuity of sum, product, etc. From the preceding article we can infer at once that the sum of two (or any finite number of) continuous functions is a continuous function, the product of two (or any finite number of) continuous functions is a continuous

function, the quotient of two functions is a continuous function except for values of x which make the denominator zero.

1·72. Method of finding limits. The propositions of the preceding article, together with the properties of the various elementary functions and the four fundamental limits evaluated in the next article, enable us to find many limits quite easily. The difficulty which is most usually met with is the simultaneous tending to zero of the numerator and denominator of a fraction. In such cases it is usually possible to divide the numerator and denominator by a common factor.

Ex. Find $\lim_{x\to 3}(x^2-4x+3)/(x^3-2x^2-3x)$.

The numerator and denominator both tend to zero as $x \to 3$. Hence theorem (iv) of § 1·7 gives us no information. But evidently $x-3$ is a factor of both the numerator and the denominator. Since the required limit depends upon the values of the given fraction for values of x other than 3, we can divide the numerator and denominator by $x-3$ and get the result:

$$\frac{x^2-4x+3}{x^3-2x^2-3x}=\frac{x-1}{x(x+1)} \text{ if } x \neq 3.$$

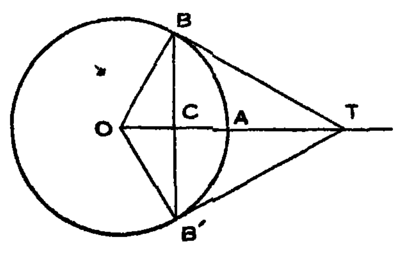
Hence the limit, as x tends to 3, of the fraction on the left is equal to the limit, as x tends to 3, of the fraction on the right. Now the fraction on the right is continuous at $x=3$ (§ 1·71). Hence the limit, as $x \to 3$, of $(x-1)/x(x+1)$ is equal to the value of $(x-1)/x(x+1)$ when $x=3$. This value is $\frac{1}{6}$. Hence

$$\lim_{x\to 3}(x^2-4x+3)/(x^3-2x^2-3x)=\tfrac{1}{6}.$$

We may note in passing that the function $(x^2-4x+3)/(x^3-2x^2-3x)$ is not defined at $x=3$.

1·73. Some Important Limits.

(i) $\mathbf{Lim}_{x\to 0}\,(\sin x)/x = 1$. Let the circular measure of each of the angles AOB, $B'OA$ be x, where $0 < x < \frac{1}{2}\pi$. Let the tangents at B and B' to the circle $B'AB$ meet OA in T and let the chord BB' meet OA in C, O being the centre of the circle $B'AB$.



We shall assume now as an axiom* that the chord $BCB' <$ the arc $BAB' < BT + TB'$.

*It is possible to *prove* this on the basis of still simpler axioms. See Hobson : *A Treatise on Plane Trigonometry*, § 92.

Dividing by OB, it follows that $\sin x < x < \tan x$,

or
$$1 < x/\sin x < 1/\cos x,$$

which is easily seen to be true also when x lies between 0 and $-\frac{1}{2}\pi$.

As $x \to 0$, $1/\cos x \to 1$. Hence $x/\sin x$ must also $\to 1$. Therefore, $\lim_{x\to 0} (\sin x)/x = 1$.

(ii) **$\mathbf{Lim}_{x\to 0}\,(1+x)^{1/x} = e$.** If n is an integer, it follows from the Binomial Theorem that

$$\left(1+\frac{1}{n}\right)^n = 1 + n\cdot\frac{1}{n} + \frac{n(n-1)}{2!}\cdot\frac{1}{n^2} + \dots$$
$$+ \frac{n(n-1)\dots(n-n+1)}{n!}\cdot\frac{1}{n^n}$$
$$= 1 + 1 + \frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots$$
$$+ \frac{1}{n!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{n-1}{n}\right). \quad . \quad (1)$$

The $(r+1)$th term is $\dfrac{1}{r!}\left(1-\dfrac{1}{n}\right)\left(1-\dfrac{2}{n}\right)\dots\left(1-\dfrac{r-1}{n}\right)$.

This is positive and increases as n increases. Hence $(1+1/n)^n$ increases as n increases, because the number of terms—each of which is positive—increases as n increases, and also every term after the second increases as n increases.

It follows that $(1+1/n)^n$ must either tend to $+\infty$, or to a definite positive limit, as $n \to \infty$ by taking up larger and larger integral values.

But, by equation (1), $(1+1/n)^n$ is less than
$$1 + 1 + 1/2! + 1/3! + \dots + 1/n!,$$
and therefore $< 1 + 1 + 1/2 + 1/2^2 + \dots + 1/2^{n-1}$, or < 3.

Therefore $(1+1/n)^n$ cannot $\to +\infty$. So it must tend to a definite positive number, usually denoted by e, where $e < 3$ and, by eqn. (1), > 2. Writing t for n, and restricting t to integral values, we have
$$\lim_{t\to\infty} (1+1/t)^t = e. \quad . \quad . \quad . \quad . \quad . \quad (2)$$

If $t \to \infty$ by taking values other than integral ones, then also the above proposition is true; because, for every value of t, we can find an integer n such that
$$n \leqslant t < n+1.$$

Taking reciprocals and adding one to each member, we get
$$1 + \frac{1}{n+1} < 1 + \frac{1}{t} \leqslant 1 + \frac{1}{n}.$$

Therefore $$\left(1+\frac{1}{n+1}\right)^n < \left(1+\frac{1}{t}\right)^t < \left(1+\frac{1}{n}\right)^{n+1},$$

or $$\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} < \left(1+\frac{1}{t}\right)^t$$
$$< \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right).$$

As $t \to \infty$, $n \to \infty$.

Hence, as $t \to \infty$, $\left(1+\frac{1}{n+1}\right)^{n+1}\left(1+\frac{1}{n+1}\right)^{-1} \to e \cdot 1$, i.e., $\to e$

Similarly also $$\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n}\right) \to e.$$

It follows that $\lim_{t\to\infty}(1+1/t)^t = e.$

We can prove also that $\lim_{t\to-\infty}(1+1/t)^t = e.$

For, if t is negative, and equal to $-u$, say,

$$\left(1+\frac{1}{t}\right)^t = \left(1-\frac{1}{u}\right)^{-u} = \left(\frac{u}{u-1}\right)^u = \left(1+\frac{1}{u-1}\right)^u$$
$$= \left(1+\frac{1}{v}\right)^v\left(1+\frac{1}{v}\right), \text{ where } v = u-1.$$

As $t \to -\infty$, $u \to \infty$ and $v \to \infty$. Taking limits, we get at once

$$\lim_{t\to\infty}(1+1/t)^t = \lim_{v\to\infty}(1+1/v)^v(1+1/v) = e. \qquad (3)$$

Putting $1/t = x$, we see from formulae (2) and (3) that

$$\lim_{x\to 0}(1+x)^{1/x} = e,$$

whether $x \to 0$ from the right or the left.

(iii) **Lim**$_{x\to 0}$ $(\mathbf{e}^x - \mathbf{1})/\mathbf{x} = \mathbf{1}$. Put $e^x - 1 = t$, so that $x = \log(1+t)$. Then

$$\lim_{x\to 0}(e^x-1)/x = \lim_{t\to 0} t/\log(1+t)$$
$$= \lim_{t\to 0} 1/\log(1+t)^{1/t} = 1/\log e = 1.$$

We can prove similarly that $\lim_{x\to 0}(a^x - 1)/x = \log a$.

(iv) **Lim**$_{x\to 0}$ $\{(\mathbf{1}+\mathbf{x})^n - \mathbf{1}\}/\mathbf{x} = \mathbf{n}$. Put $(1+x)^n - 1 = z$.

Then $$\lim_{x\to 0}\frac{(1+x^n)-1}{x} = \lim_{x\to 0}\left\{\frac{(1+x)^n-1}{\log(1+x)}\cdot\frac{\log(1+x)}{x}\right\}$$
$$= \lim_{z\to 0}\frac{z}{(1/n)\log(1+z)}\cdot\lim_{x\to 0}\log(1+x)^{1/x}$$
$$= \lim_{z\to 0}\frac{n}{\log(1+z)^{1/z}}\cdot\log e = n.$$

Examples on Chapter I

1. If $f(x) = x^3 + 2x + 1$, find $f(a^2)$.

2. If $f(x) = x^2 - x^{-2}$, show that $f(x) = -f(1/x)$.

3. If $f(x) = \dfrac{x^2+2}{x-1}$ when $x < 3$ and $f(x) = \dfrac{\sin x}{x-2}$ when $x > 3$, for what values of x is the function not defined?

4. If $f(x) = \dfrac{x^2-3x+2}{x-2}$, find the limit of $f(x)$ as x tends to 2.

5. If $f(x) = \dfrac{x^2+3x-2}{x-2}$, does $\lim_{x\to 2} f(x)$ exist? Give reasons.

6. Explain, giving suitable examples, the distinction between the value of a function $f(x)$ for $x = a$, and the limit of $f(x)$ for $x = a$.
[*Allahabad*, 1930]

7. Draw a graph of the function y given by $y = 0$ when $x = 0$ and $y = x \sin(1/x)$ when $x \neq 0$. Is this function continuous at $x = 0$?

8. If $y = x$ when x is integral and $y =$ the integral part of x when x is fractional, draw the graph of y.

9. Draw a graph of the function y defined by the statement "y is the smallest positive number that makes $x + y$ an integer". For what values of x is the function discontinuous? [*Leathem*]

10. Show that the function $\phi(x)$ which is equal to 0 when $x = 0$; to $\frac{1}{2} - x$ when $0 < x < \frac{1}{2}$; to $\frac{1}{2}$ when $x = \frac{1}{2}$, to $\frac{3}{2} - x$ when $\frac{1}{2} < x < 1$, and to 1 when $x = 1$, has three points of discontinuity which you are required to find. [*Patna*, 1937]

11. Are the following functions continuous at the origin? Explain.

(i) $f(x) = \cos(1/x)$ when $x \neq 0$; $f(0) = 0$;
(ii) $f(x) = x \sin(1/x)$ when $x \neq 0$; $f(0) = 0$;
(iii) $f(x) = x \sin(1/x)$ when $x \neq 0$; $f(0) = 2$.
[*Benares*, 1937]

CHAPTER II

DIFFERENTIATION. SIMPLE CASES

2·1. Definition. *If $f(x)$ is a function of x,*

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

is called the **differential coefficient** of *$f(x)$ for $x = a$.*

We have seen that a limit may exist or may not. The differential coefficient, therefore, may exist for $x = a$ or may not.

The differential coefficient is also called the **derivative,** or the **derived function.**

As an alternative to saying "the differential coefficient of $f(x)$ *for* $x = a$," we might say "the differential coefficient of $f(x)$ *at* $x = a$."

The process of finding the differential coefficient is called *differentiation.* We are said to *differentiate* $f(x)$. If there is any likelihood of doubt as to which symbol is the independent variable, we make it clear by saying some such thing as "differentiate $f(x)$ *with regard to* (or with respect to) x".

It should be carefully noted that in taking the above limit, $\{f(a+h) - f(a)\}/h$ is regarded as a function of h, h is regarded as a variable, and a is regarded as a constant. The sequel will show that $\lim_{h \to 0} \{f(a+h) - f(a)\}/h$ is independent of h.

Generally it is more convenient to write x itself for a, it being understood that in the process of finding the limit when $h \to 0$, x is to be kept constant. With this understanding we can write:

$$\text{the differential coefficient of } f(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}.$$

If we try to find the limit involved in the definition of a differential coefficient by actual substitution (§1·6), the numerator and denominator both become zero, and thus we get the meaningless form 0/0. We have, therefore, to employ special methods. These are given in the various articles which follow.

2·11. Notation. The differential coefficient of $f(x)$ is written generally as $\dfrac{df(x)}{dx}$, or $\dfrac{d}{dx} f(x)$, or $df(x)/dx$, or $f'(x)$, or $Df(x)$.

The differential coefficient at $x = a$ is generally represented by $f'(a)$, or $\{df(x)/dx\}_{x=a}$.

The student must not think that $df(x)/dx$ means $df(x)$ divided by dx, for $df(x)$ and dx have no meaning according to our definition. To think so would be as wrong as to think that $\log x$ is the product of log and x!

2·12. Differential coefficient of $\mathbf{x^n}$. If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Therefore, by definition,

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h}.$$

(i) Suppose first that $x \neq 0$. Then we can take out x^n common, and the right hand side can be written as

$$\lim_{h\to 0} x^n\frac{(1+h/x)^n - 1}{h}.$$

Now $x \neq 0$ by supposition. Also h tends to zero. So we may suppose h to be numerically smaller than x; i.e., h/x to be numerically smaller than unity. We can, therefore, expand $(1+h/x)^n$ by the Binomial Theorem and write the above expression as

$$\lim_{h\to 0} x^n\frac{1 + n\dfrac{h}{x} + \dfrac{n(n-1)}{1.2}\dfrac{h^2}{x^2} + \ldots - 1}{h}.$$

As h is not zero, and 1 and -1 cancel in the numerator, we can divide out by h. Thus we get $\dfrac{d}{dx}x^n$

$$= \lim_{h\to 0} x^n\left\{n\frac{1}{x} + \frac{n(n-1)}{1.2}\frac{h}{x^2} + \frac{n(n-1)(n-2)}{1.2.3}\frac{h^2}{x^3} + \ldots\right\}$$

$= \lim_{h\to 0} x^n\,\{n/x + h \times$ (a finite expression, when n is a positive integer, or, when n is not a positive integer, a convergent infinite series, the sum of which does not $\to \infty$ as $h \to 0$)$\}$

$= n\,x^{n-1}$, by theorems (i) and (iii) of § 1·7.

We have thus proved the important proposition:

$$\boxed{\frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x^n} = \mathbf{n\,x^{n-1}}.}$$

Ex. $\frac{d}{dx}x^7 = 7x^6$; $\frac{d}{dx}x^{\sqrt{2}} = (\sqrt{2})x^{\sqrt{2}-1}$; $\frac{d}{dx}x^{-5} = -5x^{-6}$.

(ii) Let $x = 0$. If $n > 0$,

$$f'(0) = \lim_{h\to 0}\frac{(0+h)^n - 0^n}{h} = \lim_{h\to 0}\frac{h^n}{h} = \lim_{h\to 0} h^{n-1}.$$

Hence, if $n > 1$, $f'(0) = 0$,

if $n = 1$, $f'(0) = 1$;

if $0 < n < 1$, $\begin{cases} f'(0) \text{ is non-existent if } h^{n-1} \text{ changes sign when} \\ \quad \text{the sign of } h \text{ is changed (cf. § 1·41),} \\ f'(0) = \infty, \text{ if } h^{n-1} \text{ does not change sign when} \\ \quad \text{the sign of } h \text{ is changed (cf. § 1·4).} \end{cases}$

If $n = 0$ and $f(0) = 1$, $f(x)$ becomes merely a constant and the case comes under § 2·17 below.

If $n < 0$, i.e., if n is negative, $f(0)$ is not defined (§ 1·12) and therefore the question of the existence of $f'(0)$ does not arise.

Alternative Proof. If we wish to avoid infinite series, we can proceed as follows :—

$$\frac{d}{dx}x^n = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} = \lim_{h\to 0} x^{n-1}\frac{(1+h/x)^n - 1}{h/x}$$

(supposing $x \neq 0$)

$$= \lim_{t\to 0} x^{n-1}\frac{(1+t)^n - 1}{t} \text{ (where } t = h/x) = x^{n-1}.\, n$$

(by § 1·73).

2·13. Differential coefficient of sin x.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \text{ by definition}$$

$$= \lim_{h\to 0}\frac{2\cos(x+\tfrac{1}{2}h)\sin\tfrac{1}{2}h}{h}$$

$$= \lim_{h\to 0}\left\{\cos(x+\tfrac{1}{2}h).\frac{\sin\tfrac{1}{2}h}{\tfrac{1}{2}h}\right\}$$

$$= \cos x,$$

because as $h \to 0$, $\cos(x+\tfrac{1}{2}h) \to \cos x$, $\cos x$ being a continuous function of x; and $(\sin\tfrac{1}{2}h)/(\tfrac{1}{2}h) \to 1$, in accordance with the well-known theorem of trigonometry (see § 1·73) that $\lim_{\theta\to 0}(\sin\theta)/\theta = 1$.

Thus $$\mathbf{\frac{d}{dx}\sin x = \cos x.}$$

2·14. Differential coefficient of cos x.

$$\frac{d}{dx}\cos x = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$

$$= \lim_{h\to 0}\frac{-2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h}$$

$$= \lim_{h\to 0}\left\{-\sin(x+\tfrac{1}{2}h)\,.\,\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right\}$$

$= -\sin x$, for reasons similar to those of the previous article.

Thus
$$\mathbf{\frac{d}{dx}\cos x = -\sin x.}$$

2·15. Differential coefficient of e^x.

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h}$$

$$= \lim_{h\to 0} e^x.\frac{e^h-1}{h}$$

$$= \lim_{h\to 0} e^x\frac{1+h+\frac{1}{2}h^2+(1/3!)h^3+\dots-1}{h}$$

$= \lim_{h\to 0} e^x\,\{1 + h \times$ (a convergent series, the sum of which does not $\to\infty$ as $h\to 0$)$\}$

$= e^x.$

Thus
$$\mathbf{\frac{d}{dx}e^x = e^x.}$$

Alternative Proof. If we wish to avoid infinite series, we can proceed as follows :—

$$\frac{d}{dx}e^x = \lim_{h\to 0}\frac{e^{x+h}-e^x}{h} = \lim_{h\to 0} e^x.\left(\frac{e^h-1}{h}\right) = 1$$

$$= e^x, \text{ because } \lim_{h\to 0}\frac{e^h-1}{h} = 1 \text{ (see § 1·73).}$$

2·16. Differential coefficient of log x, i.e., log$_e x$.

$$\frac{d}{dx}\log x = \lim_{h\to 0}\frac{\log(x+h)-\log(x)}{h}$$

$$= \lim_{h\to 0} \frac{\log(1 + h/x)}{h}$$

$$= \lim_{h\to 0} \frac{h/x - \frac{1}{2}h^2/x^2 + \frac{1}{3}h^3/x^3 - \frac{1}{4}h^4/x^4 + \dots}{h}$$

$$= \lim_{h\to 0} \{1/x - h \times \text{(a convergent series the sum of which does not} \to \infty \text{ as } h \to 0)\}$$

$$= 1/x.$$

Thus $$\frac{d}{dx} \log_e x = 1/x.$$ log$_e$ x   log$_e$ e

Alternative Proof. If we wish to avoid infinite series, we can proceed as follows :—

$$\frac{d}{dx}\log x = \lim_{h\to 0} \frac{\log(x+h) - \log(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{x} \cdot \frac{x}{h} \log\left(1 + \frac{h}{x}\right)$$

$$= \frac{1}{x} \lim_{h\to 0} \log\left(1 + \frac{h}{x}\right)^{x/h}$$

$$= (1/x) \lim_{t\to 0} \log(1 + t)^{1/t}$$

$$= (1/x) \log e \text{ (see § 1·73)} = 1/x.$$

2·17. Differential coefficient of a constant. Let c be a constant. If $f(x) = c$ for every value of x, then

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{c - c}{h} = 0 ;$$

i.e., *the differential coefficient of a constant is zero.*

2·2. Differential coefficient of the product of a constant and a function. Let a be a constant. Then

$$\frac{d}{dx}\{a f(x)\} = \lim_{h\to 0} \frac{af(x+h) - af(x)}{h}$$

$$= \lim_{h\to 0} \left\{a . \frac{f(x+h) - f(x)}{h}\right\}$$

$$= a\frac{df(x)}{dx}, \text{ (by § 1·7, Theor. iii)},$$

i.e., *the differential coefficient of the product of a constant and a function is equal to the product of the constant and the differential coefficient of the function.*

Ex. $\frac{d}{dx}(8x^7) = 8.7.x^6$; $\frac{d}{dx}\left(3x^{\sqrt{2}}\right) = 3\sqrt{2}\,x^{\sqrt{2}-1}$,

$\frac{d}{dx}(-5x^2) = -5.2.x$; etc.

2·21. Differential coefficient of $\log_a x$. We know that $\log_a x = (\log_e x).\log_a e$.

As $\log_a e$ is merely a constant, it follows from the preceding article that

$$\frac{d}{dx}\log_a x = \frac{1}{x}\log_a e.$$

2·3. Differential coefficient of a sum.

Let $$f(x) = f_1(x) + f_2(x).$$

Then $$f(x+h) = f_1(x+h) + f_2(x+h).$$

Therefore $\frac{d}{dx}f(x)$

$$= \lim_{h\to 0}\frac{\{f_1(x+h)+f_2(x+h)\} - \{f_1(x)+f_2(x)\}}{h}$$

$$= \lim_{h\to 0}\left\{\frac{f_1(x+h)-f_1(x)}{h} + \frac{f_2(x+h)-f_2(x)}{h}\right\}$$

$$= df_1(x)/dx + df_2(x)/dx.$$

It is evident that this method is applicable also to the sum of any finite number of functions. Thus

$$\frac{d}{dx}\{f_1(x) + f_2(x) + \ldots + f_n(x)\}$$
$$= \frac{d}{dx}f_1(x) + \frac{d}{dx}f_2(x) + \ldots + \frac{d}{dx}f_n(x).$$

Ex. 1. $d(8x^7 + 3\sin x)/dx = 8.7.x^6 + 3\cos x$.

Ex. 2. $d(4e^x - 5\log x)/dx = 4e^x - 5/x$.

Ex. 3. $d(\text{vers } x)/dx = d(1-\cos x)/dx = \sin x$.

2·4. Differential coefficient of the product of two functions.

Let $$f(x) = f_1(x)f_2(x).$$
Then $$f(x+h) = f_1(x+h)f_2(x+h).$$
Therefore $\frac{d}{dx} f(x)$

$$= \lim_{h\to 0} \frac{f_1(x+h)\, f_2(x+h) - f_1(x)\, f_2(x)}{h}$$

$$= \lim_{h\to 0} \frac{f_1(x+h)\,\{f_2(x+h) - f_2(x)\} + f_2(x)\,\{f_1(x+h) - f_1(x)\}}{h}$$

(by adding and subtracting one term in the the numerator)

$$= \lim_{h\to 0}\left\{ f_1(x+h)\frac{f_2(x+h) - f_2(x)}{h} + f_2(x)\frac{f_1(x+h) - f_1(x)}{h}\right\}$$

$$= f_1(x).\ df_2(x)/dx + f_2(x).\ df_1(x)/dx;$$

i.e., *the differential coefficient of the product of two functions*

= first function × diff. coeff. of second
+ second function × diff. coeff. of first.

Whenever one of the factors is merely a constant, § 2·2 should be applied.

Ex. $d(x^2 \sin x)/dx = x^2 \cos x + 2x \sin x.$

$d(e^x \cos x)/dx = -e^x \sin x + e^x \cos x.$

Examples

Write down the differential coefficients of

1. x, x^5, x^{-4}.
2. $x^{1/2}, x^{2/3}, x^{-5/6}$.
3. $\sqrt{x}, \sqrt{x^3}, \sqrt{x^{-3}}$.
4. $2x^2, 3x^6, 5x^{-7}$.
5. $8e^x, \sqrt{2}\sin x$.
6. $-3\log x, -\frac{4}{5}x^5$.
7. $4x^3 + 3\sin x$.
8. $5\cos x - 2e^x$.
9. $6\log x - \sqrt{x} - 7$.
10. $x^n + a^n$.
11. $ax^2 + 2bx + c$.
12. $(ax)^m + b^m$.
13. $x^3 \log x$.
14. $e^x \sin x$.
15. $\cos x . \log x$.
16. $\log_a x + \log x^a$.
17. $e^x \log_a x$.
18. $\sin x . \log_a x$.
19. $3x^5 e^x + 2$.
20. $9\sin x . e^x/7$.
21. $8\sqrt{x} . \log x$.
22. $1 + x + (x^2/2!) + (x^3/3!) + (x^4/4!) + \dots$

2·5. Differential coefficient of the quotient of two functions.

Let $$f(x) = f_1(x)/f_2(x).$$

Then $$f(x+h) = f_1(x+h)/f_2(x+h).$$

Therefore

$$\frac{d}{dx}\left\{\frac{f_1(x)}{f_2(x)}\right\} = \lim_{h\to 0}\left[\left\{\frac{f_1(x+h)}{f_2(x+h)} - \frac{f_1(x)}{f_2(x)}\right\}/h\right]$$

$$= \lim_{h\to 0}\frac{f_1(x+h)f_2(x) - f_2(x+h)f_1(x)}{f_2(x+h)f_2(x)\,.\,h}$$

$$= \lim_{h\to 0}$$

$$\frac{f_2(x)\left\{\dfrac{f_1(x+h) - f_1(x)}{h}\right\} - f_1(x)\left\{\dfrac{f_2(x+h) - f_2(x)}{h}\right\}}{f_2(x+h)f_2(x)}$$

(by adding and subtracting a term in the numerator, and then dividing both numerator and denominator by h)

$$= \frac{f_2(x)\dfrac{d}{dx}f_1(x) - f_1(x)\dfrac{d}{dx}f_2(x)}{\{f_2(x)\}^2};$$

i.e., *the differential coefficient of the quotient of two functions*

$$= \frac{\textbf{(Diff. coeff. of Numer.) (Denomr.)} - \textbf{(Numer.) (Diff. coeff. of Denomr.)}}{\textbf{Square of Denominator}}.$$

Ex. $\dfrac{d}{dx}\left(\dfrac{\log x}{\sin x}\right) = \dfrac{(1/x)\sin x - \log x.\cos x}{\sin^2 x}.$

2·51. Differential Coefficient of tan x. We can use the preceding formula to find the differential coefficients of $\tan x$ and $\cot x$. We have

$$\frac{d}{dx}\tan x = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right)$$

$$= \frac{d(\sin x)/dx.\cos x - \sin x.\,d(\cos x)/dx}{\cos^2 x}$$

$$= \frac{\cos x . \cos x - \sin x . (-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}.$$

Thus $$\frac{\mathbf{d}}{\mathbf{dx}} \tan \mathbf{x} = \sec^2 \mathbf{x}.$$

2·52. Differential Coefficient of cot x.

$$\frac{d}{dx} \cot x = \frac{d}{dx}\left(\frac{\cos x}{\sin x}\right)$$

$$= \frac{-\sin x . \sin x - \cos x . \cos x}{\sin^2 x}.$$

Thus $$\frac{\mathbf{d}}{\mathbf{dx}} \cot \mathbf{x} = -\operatorname{cosec}^2 \mathbf{x}.$$

EXAMPLES

Write down the differential coefficients of

1. $x^n/\log x$.
2. $x^n/\log_a x$.
3. $(\cos x)/\log x$.
4. $\dfrac{ax^2 + b}{\sin x + \cos x}$.
5. $\dfrac{\sqrt{a} - \sqrt{x}}{\sqrt{a} - \sqrt{x}}$.
6. $\dfrac{3 + \tan x}{5x + 7}$.
7. $\dfrac{\tan x + \cot x}{\log x}$.
8. $\dfrac{\cot x}{x + e^x}$.
9. $\dfrac{e^x + \tan x}{\cot x - x^n}$.
10. $\dfrac{5x^2 + 6x + 7}{2x^2 + 3x + 4}$.
11. $\dfrac{1}{\sin x}$.
12. $\dfrac{1}{\cos x}$.

2·6. Differential coefficient of a function of a function.

Consider the function $\sin x^2$. This is certainly a function of x. But its differential coefficient cannot be found by the rules given so far. We know the differential coefficient of x^2, and also of $\sin x$, but not of $\sin x^2$. In order to find the differential coefficient of $\sin x^2$, it is convenient to regard it as a function of x^2, which is itself a function of x. Thus $\sin x^2$ is regarded as a function of a function of x. We may have a function of a function of a function of...... of a function of x. Thus we may have $(\log \cos \sin \tan x^2)^n$, or something more complicated. But to begin with we shall take up only a function of a function of x.

Let $f(x) = f_1\{f_2(x)\}$.

Then $f(x+h) = f_1\{f_2(x+h)\}$.

Put $f_2(x) = t$, and $f_2(x+h) = t+k$. Then, as $h \to 0$, $f_2(x+h) \to f_2(x)$, because we suppose $f_2(x)$ to be a continuous function of x; i.e., $t + k \to t$. Hence as $h \to 0$, k also $\to 0$.

Therefore $\frac{d}{dx} f(x)$

$$= \lim_{h\to 0} \frac{f_1\{f_2(x+h)\} - f_1\{f_2(x)\}}{h}$$

$$= \lim_{h\to 0} \frac{f_1(t+k) - f_1(t)}{h}$$

$$= \lim_{h\to 0} \left\{ \frac{f_1(t+k) - f_1(t)}{k} \cdot \frac{k}{h} \right\}$$

$$= \lim_{h\to 0} \left\{ \frac{f_1(t+k) - f_1(t)}{k} \cdot \frac{f_2(x+h) - f_2(x)}{h} \right\}$$

$$= \frac{d}{dt} f_1(t) \cdot \frac{d}{dx} f_2(x), \text{ by § 1·7, Theor. iii,}$$

i.e.,
$$\frac{d}{dx} f_1(t) = \frac{d}{dt} f_1(t) \cdot \frac{dt}{dx}.$$

This important theorem is usually remembered (after dropping the suffix of f_1) in the form

$$\frac{\mathbf{df}}{\mathbf{dx}} = \frac{\mathbf{df}}{\mathbf{dt}} \cdot \frac{\mathbf{dt}}{\mathbf{dx}}.$$

The student sometimes feels a difficulty in seeing why

$$\lim_{h\to 0} \frac{f_1(t+k) - f_1(t)}{k}$$

is $df_1(t)/dt$. He should notice that as $h \to 0$, k also $\to 0$. Hence we can write the above expression as

$$\lim_{k\to 0} \frac{f_1(t+k) - f_1(t)}{k}.$$

If he now compares it with the definition given in § 2·1, he will see that we have now t in place of x and k in place of h. Therefore, as the result does not depend upon h or k, this limit must be $df_1(t)/dt$.

Ex. 1. Find the differential coefficient of $\sin x^2$.

Put $x^2 = t$. Then

$$\frac{d}{dx}\sin x^2 = \frac{d}{dx}\sin t = \frac{d}{dt}\sin t . \frac{dt}{dx}$$

$$= \cos t . \frac{dx^2}{dx} = (\cos x^2)\, 2x.$$

Ex. 2. Find the differential coefficient of $(ax^2 + b)^3$.

Put $ax^2 + b = t$. Then

$$\frac{d}{dx}(ax^2 + b)^3 = \frac{d}{dx}t^3 = \frac{d}{dt}t^3 . \frac{dt}{dx} = 3t^2 . \frac{d}{dx}(ax^2 + b)$$

$$= 3\,(ax^2 + b)^2 . (2ax).$$

Ex. 3. Find the differential coefficient of log sin x.

Put $\sin x = t$. Then

$$\frac{d}{dx}\log \sin x = \frac{d}{dx}\log t = \frac{d}{dt}\log t . \frac{dt}{dx}$$

$$= \frac{1}{t} . \frac{d}{dx}\sin x = \frac{1}{\sin x} . \cos x$$

$$= \cot x.$$

Important. It is quite easy to write down the differential coefficients of functions of a function without making substitutions as shown in the above examples. After a little practice the student should try to write down such differential coefficients at once, and only when he cannot do so he should employ substitutions.

Examples on Chapter II

Write down the differential coefficients of

1. e^{x^3}, $\sin x^3$, $\cos x^3$, $\tan x^3$, $\cot x^3$, $\log x^3$.
2. $(e^x)^3$, $\sin^3 x$, $\cos^3 x$, $\tan^3 x$, $\cot^3 x$, $(\log x)^3$.
3. e^{3x}, $\sin 3x$, $\cos 3x$, $\tan 3x$, $\cot 3x$, $\log 3x$.
4. $\log (x^n + a)$, $\log (e^x + 1)$, $\log (\sin x + 1)$, $\log \cos x$, $\log \tan x$, $\log \cot x$, $\log \log x$, $\log_a \sin x$.
5. e^{5x}, $e^{(1 + \log x)}$, $e^{\sin x}$, $e^{\cos x}$, $e^{\tan x}$, $e^{\cot x}$.
6. $\sin x^n$, $\sin (\log x)$, $\sin e^x$, $\sin (\cos x)$, $\sin (\tan x)$.
7. $\cos x^n$, $\cos (\log x)$, $\cos e^x$, $\cos (\cos x)$, $\cos (\tan x)$

8. $\tan x^5$, $\tan(\log x)$, $\tan e^x$, $\tan(\sin x)$.

9. $\sqrt{(\sin x)}$, $\sqrt{(\log x)}$, $\sqrt{(\cos x)}$, $\sqrt{(\tan x)}$, $\sqrt{(\cot x)}$.

10. $\dfrac{1}{\sin x}$, $\dfrac{1}{\log x}$, $\dfrac{1}{\cos x}$, $\dfrac{1}{x^n + a^n}$, $\dfrac{1}{\sqrt{(x+a)}}$.

11. $(ax+b)^n$, $\log(ax+b)$, e^{ax+b}, $\sin(ax+b)$, $\tan(ax+b)$.

12. $4\sin x^2 + \log(5\sin x + 6)$, $\tan e^x - 3\log(ax^4 + b)$.

13. $(\cos\sqrt{x})\log\sin x$, $\cos^4 x.\cos x^4$, $e^{\sin x}\sin e^x$.

14. $(x+a)^m(x+b)^n$, $(x^2+a)^m(x^2+b)^n$, $(x^n+a)^p(x^m+b)^q$.

15. $\dfrac{\cot x^3}{ax+b}$, $\dfrac{\tan^3 x}{ax^2+b}$, $\dfrac{\log\cos x}{\tan(\log x)}$, $\dfrac{e^{\sin x}}{\sin x^n}$, $\dfrac{\sqrt{(\sin x)}}{\sin\sqrt{x}}$.

6. $(e^x + e^{-x})/(e^x - e^{-x})$, $\log\{(ax^2+bx+c)(\sin x)(x^n+a^n)\}$, $\log\{(ax+b)/(px+q)\}$.

7. $f(x^n)$, $f(ax^n+b)$, $f(\sin x)$, $f(\tan x)$.

8. $\log x^x$, $\log(\sin x)^{\cos x}$, $\log(ax+b)^{\tan x}$.

9. $\sqrt[3]{a+bx^2}$, $\sqrt{(a+bx)^m}$, $\sqrt{a+bx^m}$.

10. $e^{a+bx-cx^2}$, $\log(a_0x^n + a_1x^{n-1} + \dots + a_n)$.

CHAPTER III

DIFFERENTIATION (*continued*). MORE DIFFICULT CASES

3·10. Differential Coefficient of a^x.

$$a^x = (e^{\log a})^x = e^{x \log a}.$$

Therefore $$\frac{d}{dx} a^x = e^{x \log a}. \log a;$$

i.e., $$\mathbf{\frac{d}{dx} a^x = a^x \log a.}$$

3·11. *Differential* Coefficient of sec x. Putting $\cos x = t$, we have

$$\frac{d}{dx} \sec x = \frac{d}{dx} (\cos x)^{-1} = \frac{d}{dx} t^{-1} = \frac{d}{dt} t^{-1}. \frac{dt}{dx}$$

$$= - 1 . t^{-2} \frac{d}{dx} \cos x = - \sec^2 x. (- \sin x).$$

Thus $$\mathbf{\frac{d}{dx} \sec x = \sec x \tan x.}$$

3·12. Differential Coefficient of cosec x.

$$\frac{d}{dx} \operatorname{cosec} x = \frac{d}{dx} (\sin x)^{-1} = - 1. (\sin x)^{-2}. \cos x.$$

Thus $$\mathbf{\frac{d}{dx} \operatorname{cosec} x = - \operatorname{cosec} x \cot x.}$$

3·13. Differential Coefficient of $\sin^{-1} x$.

Let $\sin^{-1} x = y$.

Then $x = \sin y$.

Differentiating both sides with respect to x, we get

$$1 = \frac{d}{dx}\sin y = \frac{d}{dy}\sin y . \frac{dy}{dx} = \cos y . \frac{dy}{dx}.$$

Therefore, by division, $\quad \dfrac{dy}{dx} = \dfrac{1}{\cos y}$

$$= \frac{1}{\sqrt{(1-\sin^2 y)}} = \frac{1}{\sqrt{(1-x^2)}};$$

i.e., $$\frac{d}{dx}\sin^{-1} x = \frac{1}{\sqrt{1-x^2}}.$$

It should be carefully noted that whereas $\sin^2 x$ means $(\sin x)^2$, $\sin^{-1} x$ does not mean $(\sin x)^{-1}$.

3·14. Differential Coefficient of $\cos^{-1}x$. This we can find by a procedure similar to the above, or as follows :—

By trigonometry, we have

$$\cos^{-1} x = \tfrac{1}{2}\pi - \sin^{-1} x.$$

Differentiating both sides, we get at once

$$\frac{d}{dx}\cos^{-1} x = -\frac{1}{\sqrt{1-x^2}}.$$

3·15. Differential Coefficients of $\tan^{-1}x$ and $\cot^{-1}x$.

Let $\qquad y = \tan^{-1} x.$

Then $\qquad \tan y = x.$

Differentiating both sides with respect to x, we get

$$\sec^2 y . \frac{dy}{dx} = 1. \qquad \text{(cf. § 3·13)}$$

Therefore $$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y};$$

i.e., $$\frac{d}{dx}\tan^{-1} x = \frac{1}{1+x^2}.$$

We have similarly, $$\frac{d}{dx}\cot^{-1} x = -\frac{1}{1+x^2}.$$

This also follows immediately from the equation

$$\cot^{-1} x = \tfrac{1}{2}\pi - \tan^{-1} x. \qquad (cf. \text{ § } 3·14).$$

3·16. Differential Coefficients of $\sec^{-1} x$ and $\operatorname{cosec}^{-1} x$.

Let $$y = \sec^{-1} x.$$
Then $$\sec y = x.$$
Differentiating both sides with respect to x, we get
$$\sec y \tan y \frac{dy}{dx} = 1.$$
Therefore $$\frac{dy}{dx} = \frac{1}{\sec y \tan y} = \frac{1}{\sec y \sqrt{(\sec^2 y - 1)}};$$
i.e., $$\frac{d}{dx} \sec^{-1} x = \frac{1}{x\sqrt{x^2 - 1}}.$$
Similarly, $$\frac{d}{dx} \operatorname{cosec}^{-1} x = -\frac{1}{x\sqrt{x^2 - 1}}.$$
This also follows immediately from the equation
$$\operatorname{cosec}^{-1} x = \tfrac{1}{2}\pi - \sec^{-1} x.$$

3·17. Differential Coefficient of $\operatorname{vers}^{-1} x$.

Let $$y = \operatorname{vers}^{-1} x.$$
Then $$\operatorname{vers} y = 1 - \cos y = x.$$
Differentiating, $$\sin y \frac{dy}{dx} = 1.$$
Therefore $$\frac{dy}{dx} = \frac{1}{\sin y} = \frac{1}{\sqrt{(1 - \cos^2 y)}}$$
$$= \frac{1}{\sqrt{\{1 - (1 - \operatorname{vers} y)^2\}}}$$
$$= \frac{1}{\sqrt{(2 \operatorname{vers} y - \operatorname{vers}^2 y)}};$$
i.e., $$\frac{d}{dx} \operatorname{vers}^{-1} x = \frac{1}{\sqrt{2x - x^2}}.$$
It follows immediately that
$$d\,(\operatorname{covers}^{-1} x)/dx = -1/\sqrt{(2x - x^2)}.$$

3·20. Inverse functions.

If there is a relation between x and y, we can generally express it in two ways: we can either express y as a function of x, or we can express x as a function of y. One of these functions, according to convenience, is called an inverse function. $\sin^{-1}x$, $\cos^{-1}x$, etc., are examples of such functions.

Let there be an inverse function $f^{-1}(x)$, i.e., let

$$y = f^{-1}(x),$$

where f^{-1} should be regarded as one symbol like F, or ϕ, or ψ.

When solved for x, let this become

$$x = f(y).$$

In the former function x is regarded as the independent variable and in the latter y. By differentiation we get dy/dx and dx/dy respectively. The relation between these differential coefficients can be obtained as follows :—

We have proved before that

$$\frac{df(t)}{dt} \cdot \frac{dt}{dx} = \frac{df(t)}{dx}.$$

Putting $t = y$ in this, and remembering that $x = f(y)$, we have at once

$$\frac{dx}{dy} \times \frac{dy}{dx} = 1.$$

This formula is useful in differentiating inverse functions. We could have saved a few steps in Arts. 3·13, 3·15, etc., by using it. Thus if $y = \sin^{-1} x$, then $x = \sin y$. Therefore

$$\frac{dy}{dx} = 1/\left(\frac{dx}{dy}\right) = \frac{1}{\cos y} = \frac{1}{\sqrt{(1 - \sin^2 y)}} = \frac{1}{\sqrt{(1 - x^2)}},$$

as before.

3·21. Differentiation with regard to a function. Suppose we have to differentiate $\sin x^2$ *with regard to* x^2. This means that we consider x^2 to be the independent variable. Putting $x^2 = t$, we have

diff. coeff. of $\sin x^2$ with regard to x^2

$$= \frac{d \sin x^2}{d(x^2)} = \frac{d}{dt} \sin t = \cos t = \cos x^2.$$

We can derive a general rule as follows :—

Diff. coeff. of $f(x)$ w.r.t. $\phi(x)$, i.e., $\dfrac{df(x)}{d\phi(x)}$

$$= \frac{df(x)}{dt}, \text{ where } t = \phi(x)$$

$$= \frac{df(x)}{dx} \cdot \frac{dx}{dt} \text{ by § 2·6} = \frac{df(x)}{dx} \Big/ \frac{dt}{dx} \text{ by § 3·20.}$$

Hence
$$\frac{\mathrm{df(x)}}{\mathrm{d}\phi(\mathrm{x})} = \frac{\mathrm{df(x)}}{\mathrm{dx}} \bigg/ \frac{\mathrm{d}\phi(\mathrm{x})}{\mathrm{dx}}.$$

Thus, diff. coeff. of $\sin x$ w. r. t. $\log x$

$$= \frac{d(\sin x)}{d(\log x)} = \left(\frac{d}{dx}\sin x\right) \bigg/ \frac{d}{dx}\log x = x\cos x.$$

But such cases are of no practical importance.

3·22. Generalisation of the rule for differentiating a function of a function. Suppose we have to differentiate $\log \sin x^2$. Putting $\sin x^2 = t$, we have

$$\frac{d}{dx}\log\sin x^2 = \frac{d}{dx}\log t = \frac{d}{dt}\log t \cdot \frac{dt}{dx}$$
$$= \frac{1}{t}\frac{d}{dx}\sin x^2.$$

We can now find $d(\sin x^2)/dx$ as in § 2·6.

It is clear that a similar procedure will apply in general.

In practice actual substitution can be dispensed with. A help towards this end is to note that the formula

$$\frac{df(t)}{dx} = \frac{df}{dt} \cdot \frac{dt}{dx}$$

can be written also as follows :—

$$\frac{d}{dx} f_1\{f_2(x)\} = \frac{df_1\{f_2(x)\}}{d\{f_2(x)\}} \cdot \frac{df_2(x)}{dx},$$

and therefore the mental process in differentiating $\log \sin x^2$ would be something as follows :—

We have to differentiate $\log(\sin x^2)$ with regard to x. Differentiating this with regard to $\sin x^2$ we get $1/\sin x^2$. We have to multiply this by the differential coefficient of $\sin x^2$ with regard to x. For that we differentiate it first with regard to x^2 and thus get $\cos x^2$, by which we multiply $(1/\sin x^2)$. We have now to multiply this product by the differential coefficient of x^2 with regard to x. The latter is $2x$. Hence the required differential coefficient is

$$(1/\sin x^2) \cdot \cos x^2 \cdot 2x.$$

Ex. 1. $\dfrac{d}{dx}(\log \sin x^2)^n$

$$= n\,(\log \sin x^2)^{n-1}.\ (1/\sin x^2).\ \cos x^2.\ 2x.$$

Ex. 2. $\dfrac{d}{dx} \sin e^{\cos 3\sqrt{x}}$

$$= \cos e^{\cos 3\sqrt{x}}.\ e^{\cos 3\sqrt{x}}.\ (-\sin 3\sqrt{x}).3.\ \tfrac{1}{2}\,x^{-1/2}.$$

Examples

Write down the differential coefficients of

1. $\log \sin^{-1} x^4$, $\log \cos^{-1} x^4$, $\log \tan^{-1} x^4$, $\log \sec^{-1} x^4$.
2. $(\sin^{-1} x^4)^4$, $(\cos^{-1} x^4)^n$, $(\tan^{-1} \sqrt{x})^2$, $(\cot^{-1} x^3)^{1/3}$.
3. $a^{\sin 2x}$, $a^{\tan 5x}$, $a^{\sec nx}$, $e^{\operatorname{cosec} (\sin x)}$.
4. $\sin \log(x^2+1)$, $\tan^{-1} e^{2x+1}$, $\operatorname{cosec}^{-1} a^{bx+c}$, $\sec(a^x + x^a)^2$.
5. $\sqrt{\log \sin x}$, $\sqrt{\sec \sqrt{x}}$, $\sqrt{\sin^{-1} x^5}$, $\sqrt{\cot^{-1} e^x}$.
6. $\sin^m (nx)$, $\cot^2 (3e^x + 1)$, $\operatorname{cosec}^3 (m \sin^{-1} x)$.
7. $\tan^{-1} \{x/\sqrt{(1+x^2)}\}$, $\cos^{-1} \{(x - x^{-1})/(x + x^{-1})\}$.
8. $\operatorname{cosec}^n x^m$, $(\operatorname{cosec}^{-1} x^m)^n$, $\sec^n (ax^2 + bx + c)$.
9. $\log \sin^{-1} e^{3x}$, $\log \cot^{-1} a^{5x+3}$, $\log \sec(ax+b)^3$.
10. 10^{10^x}, $\sin^{-1} (1+x^2)^{-1/2}$, $\log \cosh x$, $\log \log \log x^2$. [*Dacca*, 1936]
11. Find the differential coefficients of

    (i) $a^{\sin^{-1} x}$ with respect to $\sin^{-1} x$,

    (ii) e^t with regard to $\sqrt{t}$, [*Madras*, 1937]

    (iii) $\log_{10} x$ with regard to x^2. [*Andhra*, 1936]

3·23. Logarithmic differentiation. When a variable is raised to a variable power, neither the formula for x^n nor that for a^x is applicable. We have to take logarithms and differentiate. This process, called *logarithmic differentiation*, is also useful when the function consists of the product of a number of functions.

Ex. 1. Find the differential coefficient of $(\sin x)^{\log x}$.

Let $$y = (\sin x)^{\log x}.$$

Taking logs, $$\log y = \log x.\ \log \sin x.$$

Differentiating, $$\frac{1}{y}\cdot\frac{dy}{dx} = \log x.\frac{1}{\sin x}.\ \cos x + \frac{1}{x}\log \sin x.$$

Therefore $\dfrac{dy}{dx} = (\sin x)^{\log x}\{\log x.\ \cot x + (\log \sin x)/x\}$.

Ex. 2. Find the differential coefficient of $x^x + (\sin x)^{\log x}$. Here we cannot take logarithms directly. In such cases we have to *find the differential coefficient of each term separately.*

Let $y = x^x$. Then $\log y = x \log x$.

Therefore $\dfrac{1}{y}\dfrac{dy}{dx} = x.\dfrac{1}{x} + \log x$, or $\dfrac{dy}{dx} = x^x(1 + \log x)$.

Again let $z = (\sin x)^{\log x}$. Then $\log z =$ etc., and

$$\frac{dz}{dx} = (\sin x)^{\log x}\{\log x.\ \cot x + (\log \sin x)/x\}.$$

Therefore $$\frac{d}{dx}\{x^x + (\sin x)^{\log x}\}$$

$$= x^x(1 + \log x) + (\sin x)^{\log x}\{\log x.\cot x + (\log \sin x)/x\}.$$

3·24. The differential coefficient of $\{f_1(x)\}^{f_2(x)}$.

Let $y = \{f_1(x)\}^{f_2(x)}$. Then $\log y = f_2(x)\log f_1(x)$.

Therefore $\dfrac{1}{y}\dfrac{dy}{dx} = f_2(x).\dfrac{1}{f_1(x)}.f_1'(x) + f_2'(x).\log f_1(x)$,

or $$\frac{d}{dx}\{f_1(x)\}^{f_2(x)} = f_2(x).\{f_1(x)\}^{f_2(x)-1}f_1'(x)$$
$$+\{f_1(x)\}^{f_2(x)}f_2'(x).\ \log f_1(x);$$

i.e., *to differentiate $\{f_1(x)\}^{f_2(x)}$ differentiate first as if $f_2(x)$ were constant, then differentiate as if $f_1(x)$ were constant and add the two results.*

Ex. $$\frac{d}{dx}(\sin x)^{\log x} = (\log x)(\sin x)^{(\log x)-1}.\cos x$$
$$+ (\sin x)^{\log x}.(1/x).\log \sin x, \text{ as before.}$$

3·25. Differential Coefficient of the product of any number of functions.

Let $y = f_1(x).f_2(x).f_3(x)\ldots f_n(x)$.

Then $\log y = \log f_1(x) + \log f_2(x) + \ldots + \log f_n(x)$.

Therefore $$\frac{1}{y}\frac{dy}{dx} = \frac{f_1'(x)}{f_1(x)} + \frac{f_2'(x)}{f_2(x)} + \ldots + \frac{f_n'(x)}{f_n(x)},$$

or $\dfrac{dy}{dx} = f_1'(x)f_2(x)f_3(x)\ldots f_n(x) + f_1(x)f_2'(x)f_3(x)\ldots f_n(x) + \ldots;$

i.e., *to differentiate the product of any number of functions multiply the differential coefficient of each function taken separately by the product of all the remaining functions and add up the results.*

Examples

Differentiate

1. x^x, $x^{\sin x}$, $x^{\sin 2x}$, $x^{\cos ax}$, $x^{\cot bx}$; x^{5x^3}. [*Dacca*, 1937]

2. $x(1 - x^2)^{-1/2} \cos^{-1} x$, $xe^x \sin x$. [*Madras*, 1924, 1936]

3. $(\log x)^{\sin x}$, $(\sin x)^{\log x}$, $(\sin^{-1} x)^{\log x}$, $(\operatorname{cosec}^{-1} x)^{\log x}$.

4. $(\operatorname{vers}^{-1} x)^{\cos x}$, $(\operatorname{cosec}^{-1} x)^{\sqrt{(ax+b)}}$, $(\tan^{-1} x)^{(\cos x + \sin x)}$.

5. $(1 + 1/x)^x + x^{1+1/x}$, $(\cot x)^{\sin x} + (\tan x)^{\cos x}$. [*Agra*, 1934]

6. $\dfrac{x^3\sqrt{(x^2+4)}}{\sqrt{(x^2+3)}}$, $\sqrt{\left\{\dfrac{(x-a)(x-b)}{(x-p)(x-q)}\right\}}$, $\dfrac{x^4\sqrt{(1+\tan x)}}{\cos^2 x}$.

7. $(x-1)^2 (x+2)^3 (x+4) \log x$, $(\sin x)(\log x)x^x \cos x$.

8. $(1 - 2x)^{1/2} (1 + x)^{1/2} \sec^2 ax$, $3^x x^{5+x} \cos^{-1} x$.

3·26. Implicit functions. If the relation between and y is given by an equation involving both and this equation is not immediately solvable for y, then y is called an *implicit function* of x. On the other hand, when y is given in terms of x, y is called an explicit function of x. In the case of implicit functions there might be more than one value of y for each value of x, and there might be other complications. But, without paying heed to these, it is possible to get the value of dy/dx by mere differentiation of the given equation as it stands.

Ex. **Find** dy/dx when $x^5 + y^5 + 5xy - c = 0$.

Differentiating, we get $5x^4 + 5y^4y' + 5(xy' + y) = 0$.

Therefore $\qquad dy/dx = -(x^4 + y)/(x + y^4)$.

3·27. Parametric equations. Sometimes x and y are both expressed in terms of a third variable, usually called a *parameter*. We can always find dy/dx in such cases, without first eliminating the parameter, by the use of §§ 2·6 and 3·20.

Thus, if $x = f_1(t)$, and $y = f_2(t)$,

then
$$\frac{\mathbf{dy}}{\mathbf{dx}} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{\mathbf{dy}}{\mathbf{dt}} \Big/ \frac{\mathbf{dx}}{\mathbf{dt}}.$$

Ex. If $x = a\cos\theta$, and $y = b\sin\theta$, find dy/dx.

Here $$\frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cot\theta.$$

EXAMPLES V

Find dy/dx when

1. $x^3 + 3axy + y^3 = a^3$.
2. $x^n + y^n = a^n$.
3. $y = x^y$. [*Lucknow*, 1933]
4. $x^{2/5} + 3x^{1/5}y^{1/5} + y^{2/5} = a^{2/5}$.
5. $x^y . y^x = 1$. [*Calcutta*, 1937]
6. $e^x \log y = \sin^{-1} x + \sin^{-1} y$.
7. $x^y + y^x = a^b$.
8. $(\sin x)^{\cos y} + (\cos x)^{\sin y} = a$.
9. $x(x^2 + y^2)^{1/2} + \tan^{-1}(y/x) = 1$.
10. $\sqrt{(x^2 + y^2)} - \log(x^2 - y^2)$. [*Andhra*, 1937]
11. $x = a(t - \sin t)$, $y = a(1 - \cos t)$.
12. $x = a(\cos t + \log\tan\tfrac{1}{2}t)$, $y - a\sin t$.
13. $x = \log t + \sin t$, $y = e^t + \cos t$.
14. $x = \sin t^3 + \cos t^3$, $y = \sin t + 2\cos^{-1} t$.
15. If $x(1 + y)^{1/2} + y(1 + x)^{1/2} = 0$, prove that $dy/dx = -(1+x)^{-2}$. [*Benares*, 1933]

[Transpose, square, and solve for y. Reject the root $y = x$ which does not satisfy the given equation and differentiate the other root.]

3·28. Transformation. Sometimes an algebraic or trigonometrical transformation before differentiation materially shortens the work.

Ex. 1. Differentiate $\tan^{-1}\{2x/(1 - x^2)\}$. [*Agra*, 1930]

Put $x = \tan\theta$. Then $\dfrac{2x}{1 - x^2} = \dfrac{2\tan\theta}{1 - \tan^2\theta} = \tan 2\theta$.

Therefore $$\frac{d}{dx}\tan^{-1}\frac{2x}{1 - x^2} = \frac{d}{dx}\tan^{-1}\tan 2\theta$$

$$= \frac{d}{dx}(2\theta) = \frac{d}{dx}(2\tan^{-1}x) = \frac{2}{1 + x^2}.$$

Ex. 2. Differentiate $\dfrac{x^5 + 3x^4 + 6x^3 + 2x^2 + 6x + 2}{x^3 + 2}$.

Dividing out, we find that the given expression

$$= x^2 + 3x + 6 - 10/(x^3 + 2).$$

Differentiating, we find that the required differential coefficient

$$= 2x + 3 + \frac{10 \cdot 3x^2}{(x^3 + 2)^2}.$$

Ex. 3. Differentiate $\log \dfrac{(x^2 - 1)^{1/3}\, x^{2/3}}{ax + b}$.

The given expression $= \frac{1}{3} \log (x^2 - 1) + \frac{2}{3} \log x - \log (ax + b)$.

Therefore the required diff. coeff. $= \dfrac{2x}{3(x^2 - 1)} + \dfrac{2}{3x} - \dfrac{a}{ax + b}$.

Examples

Differentiate

1. $\log\sqrt{(x^2 + x + 1)}$, $\log \{(4x + 5)^{1/3}/(2x + 1)\}$.
2. $(x^4 + 5x^2 + 9)/(x^2 + 1)$, $(3x^3 + 7x^2 + 8x + 1)/(x^2 - 1)$.
3. $\tan^{-1} \dfrac{3a^2x - x^3}{a(a^2 - 3x^2)}$, $\tan^{-1} \dfrac{x}{\sqrt{(a^2 - x^2)}}$, $\tan^{-1} \dfrac{3ax}{a^2 - 2x^2}$.
4. $\cos^{-1} (4x^3 - 3x)$, $\cos^{-1} \sqrt{\{(1 + x)/2\}}$, $\sin^{-1} (3x - 4x^3)$.
5. $\tan^{-1} \left(\dfrac{1 - \cos x}{1 + \cos x}\right)^{1/2}$, $\tan^{-1} \dfrac{a - x}{1 + ax}$, $\tan^{-1} \dfrac{\sqrt{(1 + x^2)} - 1}{x}$.
6. $\sin^{-1} \{x\sqrt{(1 - x)} - \sqrt{x}\sqrt{(1 - x^2)}\}$. [*Agra*, 1934]
7. Find the differential coefficient of $\cot \cos^{-1} x$ and illustrate graphically. [*Bombay*, 1935]

3·3. Hyperbolic Functions. These functions are defined as follows :—

$$\cosh x = \tfrac{1}{2} (e^x + e^{-x}), \qquad \sinh x = \tfrac{1}{2} (e^x - e^{-x}),$$
$$\tanh x = \sinh x/\cosh x, \qquad \coth x = \cosh x/\sinh x,$$
$$\operatorname{sech} x = 1/\cosh x, \qquad \operatorname{cosech} x = 1/\sinh x.$$

(i) We notice that

$$\begin{aligned}\cosh^2 x - \sinh^2 x &= \tfrac{1}{4} (e^x + e^{-x})^2 - \tfrac{1}{4} (e^x - e^{-x})^2 \\ &= \tfrac{1}{4} (e^{2x} + 2 + e^{-2x}) - \tfrac{1}{4} (e^{2x} - 2 + e^{-2x}) \\ &= \tfrac{1}{4} (2 + 2);\end{aligned}$$

i.e.,
$$\cosh^2 x - \sinh^2 x = 1.$$

(ii) Again, $\dfrac{d}{dx} \sinh x = \dfrac{1}{2} \dfrac{d}{dx} (e^x - e^{-x}) = \dfrac{1}{2} (e^x + e^{-x})$;

i.e.,
$$\frac{d}{dx} \sinh x = \cosh x.$$

Similarly $$\frac{d}{dx}\cosh x = \sinh x.$$

(iii) Other formulae, which can be easily verified, are:

$$\frac{d}{dx}\tanh x = \operatorname{sech}^2 x, \qquad \frac{d}{dx}\coth x = -\operatorname{cosech}^2 x,$$

$$\frac{d}{dx}\operatorname{sech} x = -\operatorname{sech} x \tanh x, \quad \frac{d}{dx}\operatorname{cosech} x = -\operatorname{cosech} x \coth x.$$

(iv) If $y = \cosh^{-1} x$, then $x = \cosh y = \frac{1}{2}(e^y + e^{-y})$.

Multiplying by $2e^y$, and transposing, $e^{2y} - 2xe^y + 1 = 0$.

Solving as a quadratic in e^y, $e^y = x \pm \sqrt{(x^2 - 1)}$.

Therefore $$y = \log\{x \pm \sqrt{(x^2 - 1)}\}.$$

The principal value is defined as the one in which the positive sign is taken, and is the value which is usually denoted by $\cosh^{-1} x$.

Hence $$\cosh^{-1} x = \log\{x + \sqrt{(x^2 - 1)}\}.$$

Similarly $$\sinh^{-1} x = \log\{x + \sqrt{(x^2 + 1)}\},$$

$$\tanh^{-1} x = \frac{1}{2}\log\frac{1+x}{1-x}.$$

The differentiation of the inverse hyperbolic functions, therefore, would present no difficulties.

3·4. **Differentiation of Infinite Series.** Not every infinite series has a meaning, for it might be divergent or oscillatory. Even if a series is convergent and its sum is $f(x)$, the result of differentiating it term by term might give a series which does not converge to $f'(x)$. Hence, we are not justified in differentiating a series, unless we prove first that it is permissible to do so.

But these considerations are very difficult and it is sufficient for the beginner to be able formally to differentiate a series.

Similar remarks apply to continued fractions, products or other expressions in which some process is carried on an indefinitely large number of times.

Ex. If $y = x^{x^{x^{\cdots \text{ to infinity}}}}$, prove that $x\dfrac{dy}{dx} = \dfrac{y^2}{1 - y\log x}$.

[*Patna*, 1933]

We have $$y = x^y.$$

Taking logs, $$\log y = y \log x.$$

Differentiating, $$\frac{1}{y}\frac{dy}{dx} = y.\frac{1}{x} + \frac{dy}{dx}.\log x.$$

Therefore, $$\frac{dy}{dx}\left(\frac{1}{y} - \log x\right) = \frac{y}{x},$$

or $$x\frac{dy}{dx} = \frac{y^2}{1 - y\log x}.$$

3·5. Differentiation from first principles. Differentiation from first principles means that the propositions about the differentiation of sums, products, functions of functions, etc., are not to be applied; neither are the results of differentiating the standard forms to be assumed. The process is of no practical utility; but a knowledge of it is desirable. So those standard forms which have not already been differentiated from first principles and also some typical examples are considered below.

(i) a^x. The differentiation of this from first principles is very similar to the differentiation of e^x (§ 2·15). Just as e^h was expanded in powers of h, similarly a^h should be expanded. This expansion is $1 + h\log a + (h\log a)^2/2! + \dots$. The second term now is $h\log a$ instead of h. Hence the final result involves $\log a$ as a factor. If we adopt the alternative procedure of § 2·15, we must employ the result $\lim_{h\to 0}(a^h - 1)/h = \log a$. (See § 1·73.)

(ii) $$\frac{d}{dx}\tan x = \lim_{h\to 0}\frac{\tan(x+h) - \tan x}{h}$$
$$= \lim_{h\to 0}\frac{\sin(x+h)\cos x - \sin x\cos(x+h)}{h\cos(x+h)\cos x}$$
$$= \lim_{h\to 0}\frac{\sin h}{h\cos(x+h)\cos x} = \sec^2 x.$$

(iii) The differentiation of $\cot x$ is similar.

(iv) $$\frac{d}{dx}\sec x = \lim_{h\to 0}\frac{\sec(x+h) - \sec x}{h}$$
$$= \lim_{h\to 0}\frac{\cos x - \cos(x+h)}{h\cos(x+h)\cos x}$$
$$= \lim_{h\to 0}\frac{2\sin(x+\frac{1}{2}h)\sin\frac{1}{2}h}{h\cos(x+h)\cos x}$$
$$= \lim_{h\to 0}\left\{\frac{\sin(x+\frac{1}{2}h)}{\cos(x+h)\cos x}\cdot\frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\right\} = \text{etc.}$$

(v) The differentiation of $\operatorname{cosec} x$ is similar.

(vi) The differentiation of $\operatorname{vers} x$ is almost the same as that of $\cos x$.

(vii) To differentiate $\log_a x$ write it as $(\log_a e)\log x$ and proceed as in § 2·2. $\log_a e$ will remain a common factor throughout and so will appear as a factor in the result also.

(viii) To differentiate $\sin^{-1} x$, put $y = \sin^{-1} x$, and $y + k = \sin^{-1}(x + h)$. As $h \to 0$, k also $\to 0$. We have now

$$x + h = \sin(y + k),$$

and $$x = \sin y.$$

Therefore $$h = \sin(y + k) - \sin y,$$

or $$1 = \frac{\sin(y + k) - \sin y}{k} \cdot \frac{k}{h}$$

$$= \frac{\sin(y + k) - \sin y}{k} \cdot \frac{\sin^{-1}(x + h) - \sin^{-1} x}{h}.$$

Taking limits as $h \to 0$, $1 = \dfrac{d \sin y}{dy} \cdot \dfrac{d \sin^{-1} x}{dx}$.

Therefore $\dfrac{d}{dx} \sin^{-1} x = \dfrac{1}{\cos y} = \dfrac{1}{\sqrt{(1 - \sin^2 y)}} = \dfrac{1}{\sqrt{(1 - x^2)}}$.

(ix) The above method will apply in the case of all the remaining inverse circular functions.

Ex. Differentiate $\sin x^2$ from first principles.

$$\frac{d}{dx} \sin x^2 = \lim_{h \to 0} \frac{\sin(x + h)^2 - \sin x^2}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{\sin(x + h)^2 - \sin x^2}{(x + h)^2 - x^2} \cdot \frac{(x + h)^2 - x^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \frac{2 \cos \frac{1}{2}\{(x + h)^2 + x^2\} \sin \frac{1}{2}\{(x + h)^2 - x^2\}}{2hx + h^2} \times \frac{2hx + h^2}{h} \right\}$$

$$= \lim_{h \to 0} \left\{ \cos \tfrac{1}{2}\{(x + h)^2 + x^2\} \frac{\sin \frac{1}{2}(2hx + h^2)}{\frac{1}{2}(2hx + h^2)} (2x + h) \right\}$$

$$= (\cos x^2)\, 2x,$$

because as $h \to 0$, $\dfrac{\sin \frac{1}{2}(2hx + h^2)}{\frac{1}{2}(2hx + h^2)} \to 1$; $2x + h \to 2x$;

and $$\cos \tfrac{1}{2}\{(x + h)^2 + x^2\} \to \cos \tfrac{1}{2}\{x^2 + x^2\}.$$

Therefore $$\frac{d}{dx} \sin x^2 = 2x \cos x^2.$$

It would be noticed that we have here employed really a combination of §§ 2·13, 2·12 and 2·6. All such examples can be solved by a similar combination of the methods already employed for deriving the rules of differentiation and the methods employed for obtaining the differential coefficients of the standard forms.

EXAMPLES ON CHAPTER III

1. Find from first principles the differential coefficient of

$$x^n \sin ax. \qquad [Allahabad, 1923]$$

Find the differential coefficients of :—

2. $\sin^2\{\log (x^2)\}$. [*Ben.*, '30] 3. $\log (\tan^{-1} x)$ [*London*, '32]

4. $\log_{10} x$. [*London*, '34] 5. $\sqrt{\{(a^2 \;\; x^2)/(b \;\; x)\}}$.

6. $\dfrac{x}{(a^2 - x^2)^{3/2}}$ [*M. T.*, '24] 7. $\dfrac{e^{x^2} \tan^{-1} x}{\sqrt{(1 + x^2)}}$ [*Andhra*, 1936]

8. $\log_a\{(x \;\; 1)(x^2 \;\; 1)^{-1/4}\}$.

9. $\log\{\sqrt{(x \;\; a)} \;\; \sqrt{(x \;\; b)}\}$ [*Cal.*, 1936]

10. $e^{\sqrt{x+2}} - e^{\sqrt{x \;\; 2}}$ 11. $a^{\log \log x} \;\; b^{x \log x}$.

12. $\log \dfrac{\sqrt{(x + 1)} \;\; 1}{\sqrt{(x \;\; 1)} \;\; 1}$. [*Patna*, 1937]

13. $\tan^{-1}\{(b/a) \tan x\}$. [*Delhi*, 1936]

14. $\sec x^{\circ}$ [*London*, '34] 15. $\sqrt{(1 + \log x \;\; \log \sin x)}$.

16. $\log \dfrac{x^2 \;\; x \;\; 1}{x^2 - x + 1} \;\; \dfrac{2}{\sqrt{3}} \tan^{-1} \dfrac{x\sqrt{3}}{1 \;\; x^2}$ [*M. T.*, 1931]

17. $\log\{\sqrt{(1 \;\; \log x)} \;\; \sin x\}$. [*Bombay*, 1936]

18. $10^{\log \sin x}$. [*M. T.*, '32] 19. 7^{x^2+2x}. [*Madras*, '35]

20. $\cot x \coth x$. [*M. T.*, '32] 21. $(\tan x)^{\log x} + (\cot x)^{\sin x}$.

22. $\cot \cos^{-1} x$. [*Lucknow*, '33] 23. $\tan^{-1}\{(\sqrt{x})/(1 \;\; x^{3/2})\}$.

24. $\tan^{-1}\{x \sin \alpha/(1 - x \cos \alpha)\}$. [*Punjab*, 1936]

25. $\tan^{-1}\{x\sqrt{2}/(1 \;\; x^2)\}$. 26. $e^{ax} \sin bx$.

27. $\cot^3 (e^{x} \;\; x^{x})$. 28. $x \log x . \log \log x$.

29. $\sin x \sin 2x \sin 3x \sin 4x$. 30. $\tan^{-1}\left\{\dfrac{\cos x}{1 + \sin x}\right\}$.

31. $x^{\sin^{-1} x}$. [*Punjab*, '29] 32. $\text{vers}^{-1} e^{3x}$.

33. $\tan \text{arc} \sin x$. [*M. T.*, '29] 34. $x^{(\log x)^{\log \log x}}$. [*M. T.*, '28]

35. $x^{(x^x)}$. [*Mysore*, 1936] 36. $(x^x)^x$.

37. $\tan^{-1} (x^{\tan^{-1} x})$. 38. $a \cot^{-1}\{m \tan^{-1} (bx)\}$.

39. $\text{arc} \tan (\sin e^x)$. [*Dacca*, '37] 40. e^{x^n}. [*M. T.*, '24]

41. $\sqrt{t} + \cot^{-1}\sqrt{t}$, where $t = \sqrt{(x^2 + 1)}$.

42. $\log^n x$, where $\log^n$ means $\log \log \ldots$ repeated n times.

43. Express in terms of the differential coefficients of u, v, w with respect to x those of

$$u^2 v^3 w,\ u^v,\ (\log_w u)^v.$$ [*Math. Tripos*, 1925]

44. Prove that

$$\frac{d}{dx}\left\{\frac{x\sqrt{(a^2-x^2)}}{2}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right\}=(a^2-x^2)^{1/2}.$$

45. Prove that

$$\frac{d}{dx}\left\{\frac{1}{4\sqrt{2}}\log\frac{x^2+x\sqrt{2}+1}{x^2-x\sqrt{2}+1}+\frac{1}{2\sqrt{2}}\tan^{-1}\frac{x\sqrt{2}}{1-x^2}\right\}=\frac{1}{x^4+1}.$$

46. If $$\sqrt{(1-x^2)}+\sqrt{(1-y^2)}=a(x-y),$$
prove that

$$\frac{dy}{dx}=\frac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}.$$ [*Lucknow*, 1935]

Find dy/dx in the following cases:

47. $3x^4-x^2y+2y^3=0.$ [*Lucknow*, 1933]

48. $ax^2+2hxy+by^2+2gx+2fy+c=0.$ [*Patna*, 1931]

49. $(\tan x)^y+y^{\cot x}=a.$

50. $y=\cot^{-1}\{m\tan^{-1}(y/x)\}.$

51. $\left.\begin{aligned}x&=a\cos^3 t,\\ y&=a\sin^3 t.\end{aligned}\right\}$

52. $\left.\begin{aligned}x&=3at/(1+t^3),\\ y&=3at^2/(1+t^3).\end{aligned}\right\}$

53. If $\sin y=x\sin(a+y)$, prove that $\dfrac{dy}{dx}=\dfrac{\sin^2(a+y)}{\sin a}.$

54. If $x^y=e^{x-y}$, prove that $\dfrac{dy}{dx}=\dfrac{\log x}{(1+\log x)^2}.$ [*Benares*, 1934]

55. Differentiate

$\tan^{-1}[\{(1+x^2)^{1/2}+(1-x^2)^{1/2}\}\{(1+x^2)^{1/2}-(1-x^2)^{1/2}\}^{-1}].$

[Put this expression equal to y and show that

$$(\tan y+1)/(\tan y-1)=(1+x^2)^{1/2}/(1-x^2)^{1/2}.$$

Square and solve for x^2, which would be found to be equal to $\sin 2y$. Now differentiate.]

56. Differentiate $\tan^{-1}\dfrac{x^{1/3}+a^{1/3}}{1-a^{1/3}x^{1/3}}.$

57. Differentiate $\tan^{-1}[\{\sqrt{(1+x^2)}-1\}/x]$ with respect to $\tan^{-1}x.$ [*Patna*, 1934]

58. Differentiate $(\log x)^{\tan x}$ with regard to $\sin(m\cos^{-1}x).$

59. Differentiate $\tan^{-1}\{2x/(1-x^2)\}$ with regard to $\sin^{-1}\{2x/(1+x^2)\}.$ [*Benares*, 1935]

Find dy/dx if:

60. $y = \frac{x}{a+}\,\frac{x}{b+}\,\frac{x}{a+}\,\frac{x}{b+}\,\frac{x}{a+}\,\frac{x}{b+}\cdots$ to inf.

61. $y = (\cos x)^{(\cos x)^{(\cos x)\cdots\text{to inf.}}}$

62. $y = (1+x)\left(1-\frac{x^2}{2}\right)\left(1+\frac{x^3}{3}\right)\left(1-\frac{x^4}{4}\right)\cdots$ to inf.

63. $y = \sqrt{\tan x + \sqrt{\tan x + \sqrt{\tan x + \cdots \text{to inf.}}}}$

64. If $y = x + \frac{1}{x+}\,\frac{1}{x+}\,\frac{1}{x+}\cdots$ to inf., prove that

$$\frac{dy}{dx} = \frac{1}{2-}\,\frac{x}{x+}\,\frac{1}{x+}\,\frac{1}{x+}\,\frac{1}{x+}\cdots \text{ to inf.}$$

65. If $\frac{P}{Q} = a + \frac{1}{a_1+}\,\frac{1}{a_2+}\cdots\frac{1}{a_n+x}$, prove that $\frac{d}{dx}\left(\frac{P}{Q}\right) = \pm\frac{1}{Q^2}$.

[*Coll. Ex.*]

66. Given that

$$\cos\frac{x}{2}\cos\frac{x}{2^2}\cos\frac{x}{2^3}\cos\frac{x}{2^4}\cdots\cdots \text{ ad inf.} = \frac{\sin x}{x},$$

prove that $\frac{1}{2}\tan\frac{x}{2} + \frac{1}{2^2}\tan\frac{x}{2^2} + \frac{1}{2^3}\tan\frac{x}{2^3} + \cdots$ ad. inf. $= \frac{1}{x} - \cot x$.

and $\frac{1}{2^2}\sec^2\frac{x}{2} + \frac{1}{2^4}\sec^2\frac{x}{2^2} + \frac{1}{2^6}\sec^2\frac{x}{2^3} + \cdots$ ad inf. $= \operatorname{cosec}^2 x - \frac{1}{x^2}$.

CHAPTER IV

SIMPLE APPLICATIONS

4·1. **Velocity.** The student must be familiar with the idea of velocity. Suppose now that a particle is falling under gravity, so that the velocity of the particle goes on increasing. What exactly is meant by the velocity at a particular instant?

Assume that the particle started from A at the instant from which we reckon time.

Assume further that at time t the particle is at B, and let $AB = s$. What is the velocity at time t?

Suppose we wait till the time has become $t + h$. Let the particle then be at C, BC being equal to k.

Then k/h is the average velocity during the interval h under consideration.

But this average velocity must be greater than the velocity at the beginning of the interval, because the velocity is increasing. If we take h to be very small, the velocity at time t would be *very nearly* equal to k/h, but *not exactly*. This difficulty would remain however small h may be taken, provided it is not made zero.

But if we make h zero, then k also becomes zero, and we cannot infer what the velocity is.

So we have to define velocity as the limit of k/h when $h \to 0$.

Now, if we regard t as an independent variable, we may write $s = f(t)$.

$$\text{So } \lim_{h\to 0} \frac{k}{h} = \lim_{h\to 0} \frac{s + k - s}{h}$$

$$= \lim_{h\to 0} \frac{f(t+h) - f(t)}{h} = \frac{df(t)}{dt} = \frac{ds}{dt}.$$

Thus, $$\textbf{velocity} = \frac{\mathbf{ds}}{\mathbf{dt}}.$$

We may expect, therefore, that the differential calculus would be very useful in mechanics.

4·11. Acceleration. If v is the velocity at time t, considerations similar to the above will lead us to define acceleration as follows:

$$\textbf{acceleration} = \frac{\mathbf{dv}}{\mathbf{dt}}.$$

Ex. A particle is projected downwards with a velocity u and the acceleration is constant and equal to f. Find the velocity at any instant and the space described in time t.

Here $$\frac{dv}{dt} = f.$$

We can guess, therefore, that

$$v = ft + A, \quad \ldots\ldots\ldots \quad (1)$$

where A is a constant.

[The student might wonder why $ft + A$ has been taken instead of simply ft; the latter value of v does make dv/dt equal to f as required. But he must remember that ft is merely a particular case of $ft + A$, viz., the case when A is zero, and a particular case might not serve our purpose.]

It is given that the velocity is equal to u when $t = 0$. Substituting these values of v and t in (1) we get

$$u = f\cdot 0 + A,$$

which gives the value of A. Thus (1) becomes

$$\mathbf{v = u + ft.}$$

But $v = ds/dt$. Hence

$$\frac{ds}{dt} = u + ft.$$

We can guess, therefore, that

$$s = ut + \tfrac{1}{2}ft^2 + B.$$

But $s = 0$ when $t = 0$. Hence $B = 0$. We have therefore

$$\mathbf{s = ut + \tfrac{1}{2}ft^2.}$$

4·2. Increments and their ratios. If x is 3 at first, and then becomes 4, we say that there is an *increment* of 1 in x. If, however, x is 3 at first and becomes 2 afterwards, there is a diminution in x; but it is more convenient to say that there is an increment of -1 in x.

In general, if x changes from x_1 to x_2, the increment in x is $x_2 - x_1$, whether $x_2 - x_1$ be positive or negative.

The increment in x has so far been denoted by h, and the increment in y by k. But in the applications of the differential calculus it is more convenient to denote the increment in x by δx, the increment in y by δy, and so on.

The reason is that if there are many quantities which change and we use letters like h, k, l, ... to denote their increments, it becomes difficult to remember which symbol is the increment of a particular quantity. But the student must be careful to regard δx as one symbol, and not as the product of δ and x.

We shall now show that *the differential coefficient of a function y of x with respect to x is the limit of the ratio of corresponding increments in y and x, as the increment in x tends to zero* (positive and negative increments both being considered).

If y is a function of x, say $y = f(x)$, and x, y and $x + h, y + k$ are pairs of corresponding values, then the ratio of the increment in y to that in x is k/h.

$$\text{Now } \lim_{h\to 0} \frac{k}{h} = \lim_{h\to 0} \frac{y + k - y}{h}$$

$$= \lim_{h\to 0} \frac{f(x + h) - f(x)}{h},$$

i.e.,
$$\lim_{h\to 0} \frac{k}{h} = \frac{dy}{dx},$$

which proves the proposition.

In the usual notation for increments,

$$\lim_{\delta x\to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}.$$

4·3. Rate of Increase. Consider the problem of § 4·1 again, in which s is the distance described by a particle in time t. The average rate of increase of s in an interval of time δt

$$= \frac{\text{increase in } s \text{ in the interval } \delta t}{\delta t}, \text{ i.e., } \frac{\delta s}{\delta t}.$$

Therefore the rate of increase of s at time t

$$= \lim_{\delta t \to 0} \frac{\delta s}{\delta t} = \frac{ds}{dt} \text{ (by § 4·2).}$$

So we can assign a new meaning to the differential coefficient ds/dt. It is the rate at which s increases with t.

But it is not necessary that the independent variable be time. We can conceive of the rate (at a given instant) of growth of y with x.

The differential coefficient dy/dx will then be the rate at which y increases with x.

Ex. A balloon, which always remains spherical, has a variable radius. Find the rate at which its volume is increasing with the radius when the latter is 10 inches.

If the radius is x, the volume (say y) is $\frac{4}{3}\pi x^3$.

Hence the rate at which the volume increases with x is

$$\frac{d}{dx}(\tfrac{4}{3}\pi x^3), \text{ i.e., } \tfrac{4}{3}\pi . 3x^2.$$

Hence the required rate of increase $= \frac{4}{3}\pi . 3 . 10^2$ cu. in. per unit (1 in.) increase in the radius.

[*Verification.* We can verify the above by elementary methods. Suppose the radius becomes 10 inches + 0·01 inch.

Then the new volume is $\frac{4}{3}\pi(10·01)^3$ cu. in.

The old volume was $\frac{4}{3}\pi\, 10^3$ cu. in.

Hence the increase in the volume $= \frac{4}{3}\pi(10·01^3 - 10^3)$ cu. in.

The increase in the radius $= 0·01$ in.

Thus the average rate of increase of volume with radius (in cubic inches per inch)

$$= \tfrac{4}{3}\pi . \frac{10·01^3 - 10^3}{0·01}$$

$$= \tfrac{4}{3}\pi\{10^3 + 3.10^2.(0·01) + 3.10.(0·01)^2 + (0·01)^3 - 10^3\}/0·01 \text{ by the Binomial Theorem}$$

$$= \tfrac{4}{3}\pi\,\{3 . 10^2 + 3 . 10 . (0{\cdot}01) + (0{\cdot}01)^2\}$$
$$= \tfrac{4}{3}\pi . . 3 . 10^2 \text{ nearly,}$$

which is the same as before. Hence we have approximately verified the calculation made above.

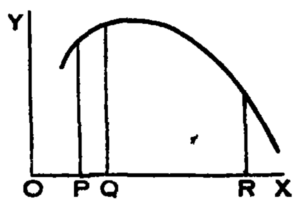
By supposing the increase in the radius to be h instead of $0{\cdot}01$, the average rate would have come out as

$$\tfrac{4}{3}\pi\,\{3 . 10^2 + 3 . 10 . h + h^2\},$$

and taking the limit as $h \to 0$, we can easily show that the *exact* (and not merely an approximate) rate is $\tfrac{4}{3}\pi . 3 . 10^2$.]

4·31. Meaning of the sign of the Differential Coefficient. Let $f(x)$ be a function of x. When $x = a$, the function has the value $f(a)$. If x is given a small positive increment, $f(x)$ might become greater than $f(a)$, or might not. If it does, we may express this fact by saying that *$f(x)$ increases with x at $x = a$.*

We took the increment in x to be *small*. If we take a large increment (say PR instead of PQ in the marginal figure), the new value of the function might be less than before. When we say that $f(x)$ increases with x at $x = a$, we mean merely that we can find a number h_1 (i.e., PQ in the figure) such that $f(a + h) > f(a)$ when $0 < h < h_1$. This is sometimes expressed by saying that $f(a + h) > f(a)$ for *sufficiently small (positive) values of h.*



We can now show that *if for any value, say a, of x the differential coefficient dy/dx is positive, it means that $f(x)$ increases with x at $x = a$. On the other hand, if dy/dx is negative at $x = a$, it means that $f(x)$ decreases as x increases.*

For, if dy/dx is positive at $x = a$,

$$\frac{f(a + h) - f(a)}{h}$$

must be positive, at least for sufficiently small values of h. This means that $f(a + h) - f(a)$ must be positive for sufficiently small positive values of h, i.e., $f(x)$ must increase with x at $x = a$.

Similarly, if dy/dx is negative at $x = a$, $f(x)$ must decrease as x increases.

The student should note that "$f(x)$ decreases with x" means that $f(x)$ decreases as x *decreases*, and so he must not use this expression when he means that "$f(x)$ decreases as x increases."

We shall show in a later chapter what happens when dy/dx is zero and is thus neither positive, nor negative We shall show that at such a point $f(x)$ has got in general either a maximum (greatest) or minimum (least) value.

Ex. 1. Show that the function x^3 steadily increases from $x = -\infty$ to $x = +\infty$, but x^4 decreases from $x = -\infty$ to $x = 0$ and then increases.

If $y = x^3$, $dy/dx = 3x^2$, which is always positive. Hence for every value of x, x^3 increases as x increases

Again if $y = x^4$, $dy/dx = 4x^3$, which is negative when x is, negative, and positive when x is positive, which shows that for negative values of x, x^4 diminishes as x increases (algebraically, of course), but for positive values of x, x^4 increases with x.

[The student should draw graphs of these functions].

Ex. 2 The top of a ladder 20 feet long is resting against a vertical wall on a level pavement, when the ladder begins to slide outward. At the moment when the foot of the ladder is 12 feet from the wall it is sliding away from the wall at the rate of 2 feet per second. How fast is the top sliding downwards at this instant? How far is the foot from the wall when it and the top are moving at the same rate?

At time t seconds reckoned from some fixed instant let the distance of the foot of the ladder from the wall be x feet and the height from the pavement of the top of the ladder be y feet. Then, by geometry,

$$x^2 + y^2 = 20^2. \qquad \text{. . . . . (1)}$$

Differentiating with respect to t, we have

$$x\frac{dx}{dt} + y\frac{dy}{dt} = 0. \qquad \text{. . . . . (2)}$$

At the moment mentioned in the problem, $x = 12$, $dx/dt = 2$. So $y = 16$ by (1), and $dy/dt = -\frac{3}{2}$ by (2)

Hence the top of the ladder is sliding *downwards* at the rate of $1\frac{1}{2}$ feet per second.

Again, if dx/dt and dy/dt are equal numerically, (2) shows that x must be equal to y. Hence by (1) the value of x then is $10\sqrt{2}$; i.e., the foot of the ladder is at a distance of $10\sqrt{2}$ feet from the wall when it and the top are moving at the same rate.

EXAMPLES

1. Prove that if a particle moves so that the space described is proportional to the square of the time of description, the velocity will be proportional to the time, and the rate of increase of the velocity will be constant. [*Dacca*, 1938]

2. A point moves in a fixed straight path so that

$$s = \sqrt{t},$$

show that the acceleration is negative and proportional to the cube of the velocity.

3. If the path of a moving point is the sine curve

$$x = at, \; y = b \sin at,$$

show (i) that the x component of the velocity is constant, and (ii) that the acceleration of the point at any instant varies as the distance from the axis of x.

4. Show that the volume of a spherical soap bubble increases $4\pi r^2$ times as fast as the radius.

5. Sand is being poured on the ground from the orifice of an elevated pipe, and forms a pile which has always the shape of a right circular cone whose height is equal to the radius of the base. If sand is falling at the rate of a cubic feet per second, how fast is the height of the pile increasing when the height is b feet?

6. A point source of light is hung a feet directly above a straight horizontal path on which a boy b feet in height is walking. How fast is the boy's shadow lengthening when he is walking away from the light at the rate of c feet per minute?

7. A rod AB, 10 feet long, moves with its ends A and B on two perpendicular lines OX and OY respectively. If A is 8 feet from O and is moving away at the rate of 2 feet per second, find at what rate the end B is moving. [*Madras*, 1936]

8. An inverted cone has a depth of 10 cm and a base of radius 5 cm. Water is poured into it at the rate of $1\frac{1}{2}$ c.c. per minute. Find the rate at which the level of the water in the cone is rising when the depth is 4 cm. [*Madras*, 1937]

9. A particle describes an ellipse whose semi-axes are 4 feet and 3 feet with a constant speed of 1 foot per second. Find the velocity of the foot of the perpendicular from the particle on the major axis, when the particle is at a distance of 1 foot from the major axis. [*Madras*, 1934]

10. Prove that as x increases,

$$(a \sin x + b \cos x)/(c \sin x + d \cos x),$$

where a, b, c, d are constants, either increases for all values of x, or decreases for all values of x. [*Andhra*, 1937]

11. Find the range of values of x for which the function $x^3 - 6x^2 - 36x + 7$ increases with x. [*Madras*, 1937]

4·4. Approximate Calculations. Small increments.

Since $$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx},$$

we must have, for small values of δx,

$$\frac{\delta y}{\delta x} = \frac{dy}{dx} \text{ approximately};$$

i.e., $$\delta y = \left(\frac{dy}{dx}\right)\delta x \textit{ approximately.}$$

This formula is very useful in calculating small changes, and is of immense importance in the theory of small errors, in physics, and in several other branches of science. It will be shown later that we can safely employ this formula whenever δx is so small that its second and higher powers can be neglected.

NOTE. δx is the **absolute error** in x. The **relative error** in x is $\delta x/x$. The **percentage error** is $100(\delta x/x)$. Thus if an error of $0''{\cdot}01$ is committed in measuring a length of $5''$, the absolute error is $0''{\cdot}01$, the relative error is $0{\cdot}002$ and the percentage error is $0{\cdot}2$.

Ex. 1. A balloon is spherical and has a radius of 10 inches. If its radius increases by $0{\cdot}1$ per cent, find approximately the percentage increase in the volume.

The volume $= V$, say, $= \frac{4}{3}\pi r^3$, where r is the radius.

By logarithmic differentiation,

$$\frac{1}{V}\frac{dV}{dr} = \frac{3}{r}.$$

Hence $$\frac{\delta V}{V} = \frac{3}{r}\,\delta r \text{ approximately.}$$

Multiplying both sides by 10 and putting $0{\cdot}1$ for $100\,\delta r/r$, we have at once:

the percentage increase in $V = 3 \times 0{\cdot}1$ approximately.

Ex. 2. Given $\log_e 4 = 1{\cdot}3863$, find $\log_e 4{\cdot}01$.

$$\delta(\log x) = (1/x)\,\delta x = \tfrac{1}{4} \times 0{\cdot}01 = 0{\cdot}0025.$$

Therefore $\log_e 4{\cdot}01 = 1{\cdot}3863 + 0{\cdot}0025 = 1{\cdot}3888$.

EXAMPLES

1. Given $\log_{10} 4 = 0{\cdot}6021$, calculate approximately $\log_{10} 404$, it being given that $\log_{10} x = 0{\cdot}4343 \log_e x$. [*Andhra*, 1936]

2. Given that $\log_{10} e = 0{\cdot}4343$, find $\log_{10} 10{\cdot}1$.

3. A regular hexagon $ABCDEF$ consists of six equal rods which are each of weight W pounds and length a feet and are freely jointed together. The hexagon rests in a vertical plane with AB in contact with a horizontal table. Prove that the decrease in potential energy when the angles A and B each increase by $1°$ is approximately

$$3aW \sin 1° \quad \text{foot-pounds.}$$

[Note that the measure of $1°$ in radians is equal to $\sin 1°$ approximately.]

4. A gas expands isothermally. If the volume v and the pressure p are connected by the relation $pv =$ constant, find the increase in pressure when the volume of a quantity of gas under a pressure of 20 lbs per square inch is altered from 1000 cubic inches to 999 cubic inches.

5. A tower is at a distance of 500 feet, and its top is observed to be at an elevation of $30°$. Calculate the total height of the tower, supposing the height of the observer to be 5 feet. If the angle of elevation was really $30° \, 12'$, what error has crept into the calculated height of the tower? Given $12' = 0{\cdot}0035$ radian approximately.

6. The radius of a sphere is found by measurement to be $18{\cdot}5$ inches with a possible error of $0{\cdot}1$ inch. Find the consequent errors possible in (1) the surface area, (2) the volume, as calculated from this measurement. [*Madras*, 1936]

7. The pressure p and the volume v of a gas are connected by the relation $pv^{1{\cdot}4} =$ constant. Find the percentage increase in the pressure corresponding to a diminution of $\frac{1}{2}$ per cent in the volume. [*Andhra*, 1937]

8. The time T of a complete oscillation of a simple pendulum of length l is given by the equation $T = 2\pi\sqrt{(l/g)}$, where g is a constant. Find the approximate error in the calculated value of T corresponding to an error of 2 per cent in the value of l. [*Madras*, 1937]

9. With the usual notation, if $\triangle$ be the area of a triangle, prove that the error in $\triangle$ resulting from a small error in the measurement of c, is given by

$$\delta\triangle = \tfrac{1}{4}\triangle\{1/s + 1/(s-a) + 1/(s-b) - 1/(s-c)\}\delta c.$$

[*Madras*, 1934]

4·5. Applications to the Theory of Equations. If $f(x)$ is a rational integral algebraic function of x and the equation $f(x) = 0$ has two roots equal to a, then $f(x)$ must be of the form

$$(x-a)^2 F(x).$$

By actual differentiation we see that the equation $f'(x) = 0$ must in this case be

$$2(x-a)\,F(x)+(x-a)^2\,F'(x)=0\,;$$

so that a is also a root of $f'(x)=0$.

It is easy to show similarly that *whenever a is a multiple root of $f(x)=0$, it will also be a root of $f'(x)=0$.* Thus, by finding the G. C. M. of $f(x)$ and $f'(x)$, we can find such roots.

Ex. The equation $x^4-x^3-x+1=0$ has one double root. Find it and hence solve the equation.

$$\frac{d}{dx}(x^4-x^3-x+1)=4x^3-3x^2-1.$$

The G.C.M. of $4x^3-3x^2-1$ and x^4-x^3-x+1 is $x-1$.
Hence $(x-1)^2$ is a factor of x^4-x^3-x+1.
By division we find the other factor to be x^2-x+1.
This equated to zero gives the roots as $-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$.
Hence the four roots of $x^4-x^3-x+1=0$ are

$$1,\ 1,\ -\tfrac{1}{2}+\tfrac{\sqrt{3}}{2}i, \text{ and } -\tfrac{1}{2}-\tfrac{\sqrt{3}}{2}i.$$

4·51. Approximate solutions of equations. Newton's Method.* If we have to get a better value of a root of the equation $f(x)=0$, when it is known that its approximate value is a, the proposition of § 4·4 is useful.

For, if $a+\delta x$ be the root, then $f(a+\delta x)$ must be zero.

Also, as a is an approximate value of the root, δx must be small. So by § 4·4.

$$\delta f(x)=f'(x)\,\delta x \text{ nearly,}$$

i.e., $$f(a+\delta x)-f(a)=f'(a).\,\delta x, \text{ nearly.}$$

Hence we have, since $f(a+\delta x)=0$,

$$\delta x=-\mathbf{f(a)/f'(a)}, \text{ approximately.}$$

This determines δx, which added to a gives a better value of the root.

It is obvious that $f(x)$ need not be algebraical.

NOTE. Taking $a+\delta x$ as the new approximate value of the root, the above process can be repeated to find a still better value of the root, and so on.

Ex. Find that root of the equation

$$x^4-12x+7=0$$

which is approximately equal to 2.

*Named after Isaac Newton (1642-1727), the great English mathematician, who was the first to invent the calculus (see the Historical Note at the end of this book). The method given here is the modification originally given by Joseph Raphson (1648-1715), a Fellow of the Royal Society, London. Hence this method is also called the "Newton-Raphson Method."

Here $f'(x) = 4x^3 - 12.$

So $f'(2) = 20.$

Also $f(2) = -1.$

Therefore $\delta x = +1/20 = 0{\cdot}05.$

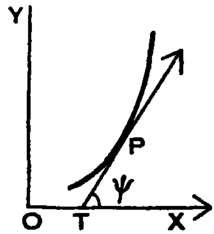
Hence the root is approximately equal to $2{\cdot}05$.

[Repetition of the process a few times would show that the root is $2{\cdot}047275567$ correct to 9 decimal places. The method, however, becomes very laborious after one or two steps.]

4·6. Applications to Geometry. Another meaning of the Differential Coefficient. The differential coefficient is useful in evaluating limits which cannot be found by mere substitution (Chap. XIV), in finding maxima and minima of functions (Chap. XIII) and in expanding functions of x in powers of x (Chap. VI).

In geometry it is useful in finding tangents and normals of curves (Chaps. VII and VIII), the rate at which they bend (Chap. IX) and their shapes (Chap. X).

These various topics will be dealt with in due course. In particular, it will be shown that if $y = f(x)$ be a curve, then the tangent to it at any point (x, y) makes with the axis of x an angle ψ whose trigonometrical tangent is dy/dx; i.e.,



$$\tan \psi = \frac{dy}{dx}.$$

This gives us a new meaning of the differential coefficient, which is of great importance in the application of the differential calculus to Geometry.

EXAMPLES ON CHAPTER IV

1. If at time t the displacement s of a particle moving away from the origin is given by

$$s = a \sin t + b \cos 2t,$$

find the velocity and acceleration of the particle.

2. A stone is dropped into a calm lake, sending out a series of concentric ripples. If the radius of the outer ripple increases

uniformly at the rate of c feet per second, how rapidly is the disturbed area increasing t seconds after the stone hits the water?

3. A military observer in an aeroplane is ascending at the rate of a miles an hour. How fast is the visible area of the earth's surface increasing in square miles per minute t minutes after the plane left the ground? (The radius of the earth may be assumed to be r miles.)

4. Van der Waal's equation for a gas is

$$(p + av^{-2})(v - b) = k,$$

where p is the pressure, v the volume, and a, b and k are constants. What is the change in volume per unit increase in the pressure?

5. If the height of a cone increases by a per cent, its semi-vertical angle remaining the same, what is the approximate percentage increase (i) in the total area and (ii) in the volume, assuming that a is small?

6 The angle of incidence ϕ and of refraction ψ are connected by the relation $\sin\phi = \mu\sin\psi$, where μ is a constant. If ϕ should change from $60°$ to $59°\ 50'$, what would be the corresponding change in the value of ψ, supposing that its former value was $45°$?

7 The area of a triangle is calculated from the angles A and C and the side b If a small error δA is made in measuring A, show that the percentage error in the area is approximately

$$100\,\delta A \sin C/\{\sin A \sin (A + C)\}$$ [*London*]

8. The rate of flow Q of water per second over a sharp-edged notch of length l, the height of the general level of the water above the bottom of the notch being h, is given by the formula

$$Q = c(l - h/5)h^{3/2},$$

where c is a constant.

Show that for a small error δh in the measurement of h, the error δQ in Q is

$$\tfrac{1}{2}c(3l - h)h^{1/2}.\ \delta h.$$

9. If $f'(x) > 0$, for $a \leqslant x \leqslant b$, show that $f(x)$ is an increasing function of x in this interval.

Two points A and B lie on a fixed straight line. They are on opposite sides of, and at equal distances from, a fixed point O on the line. P is any fixed point not on the line AOB Show that the sum of the distances of P from the points A and B is increased, if the distance AB is increased. [*Math Tripos*, 1934]

10. Solve the following equations by finding first their multiple roots:

(i) $4x^4 - 27x^2 - 25x - 6 = 0,$

(ii) $x^4 + 3x^3 - 3x^2 - 7x + 6 = 0,$

(iii) $8x^4 - 20x^3 + 18x^2 - 7x + 1 = 0.$

11. What condition must a and b satisfy in order that the equation $x^3 + x^2 + ax + b = 0$ may have a double root.

12. One root of the equation

$$x^4 - 12x^2 - 12x - 3 = 0$$

is approximately equal to 4. Find it correctly to two decimal places.

13. Show by starting from the rough approximation $x = \pi$ as a root of the equation $\sin x = ax$, where a is small, that a much better approximation can be obtained as

$$x = (1 - a + a^2)\pi. \qquad \text{[\textit{Dacca}, 1936]}$$

14. The motion of the needle of a galvanometer is given by the equation $\theta = 6e^{-t/2} \sin 3t$, where θ is the angle in radians made by the needle with the zero position at the end of t seconds. Find the angular velocity of the needle at time t, and show that the extreme excursions to the right and left of the zero position occur at intervals of $\pi/3$ seconds, and that the angles corresponding to these extreme excursions form a geometrical progression of common ratio $-e^{-\pi/6}$. [*Madras*, 1936]

15. The angle A of a triangle ABC is found by measurement to be 63 degrees, and the area is calculated by the formula $\frac{1}{2}bc \sin A$. Find the percentage error in the calculated value of the area, due to an error of 15 minutes in the measured value of A. [*Madras*, 1937]

16. A circle is drawn with its centre on a given parabola and touching its axis. Show that if the point of contact recede with a constant velocity from the vertex of the parabola, the rate of increase of the area of the circle is also constant. [*Madras*, 1936]

17. $ABCD$ is a rectangular protractor in which $AB = 6$ inches, $BC = 2$ inches and O is the midpoint of AB. An angle BOP is indicated by a mark P on the edge CD. If in setting off an angle θ degrees, a mark is made $1/100$ of an inch along the edge from the correct spot, show that the error in the angle is

$$9 \sin^2 \theta / 10\pi \text{ degrees.} \qquad \text{[\textit{Andhra}, 1937]}$$

CHAPTER V

SUCCESSIVE DIFFERENTIATION

5·1. Definition and Notation. If y be a function of x, its differential coefficient dy/dx will be in general a function of x which can be differentiated. The differential coefficient of dy/dx is called the second differential coefficient of y. Similarly, the differential coefficient of the second differential coefficient is called the third differential coefficient, and so on. The successive differential coefficients of y are denoted by

$$\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots,$$

the nth differential coefficient of y being $\dfrac{d^ny}{dx^n}$.

Alternative methods of writing the nth differential coefficient are

$$\left(\frac{d}{dx}\right)^n y,\ D^n y,\ y_n,\ d^ny/dx^n,\ y^{(n)}.$$

In the last case the first, second, etc., differential coefficients would be written as y', y'', y''', etc.

The value of a differential coefficient at $x = a$ is usually indicated by adding a suffix; thus: $(y_n)_{x=a}$ or $(y_n)_a$. If $y = f(x)$, the same thing can also be indicated by $f^{(n)}(a)$.

5·2. Standard Results.

(1) If $y = (ax + b)^m$, then $y_1 = m \cdot a(ax + b)^{m-1}$, $y_2 = m(m - 1) \cdot a^2(ax + b)^{m-2}$, etc. In general

$$\mathbf{D^n (ax + b)^m}$$
$$\mathbf{= m(m - 1)(m - 2) \ldots (m - n + 1)\, a^n (ax + b)^{m-n}.}$$

If m is a positive integer, the $(m + 1)$th and all the successive differential coefficients would be zero.

(2) If $y = e^{ax}$, then $y_1 = ae^{ax}$, $y_2 = a^2$. In that the general

$$\mathbf{D}^n \, e^{ax} = a^n \, e^{ax}.$$

(3) Similarly $\mathbf{D}^n \, a^x = (\log a)^n \, a^x.$

(4) If $y = \log(ax + b)$, $y_1 = a(ax + b)^{-1}$, $y_2 = (-1)a^2(ax + b)^{-2}$, $y_3 = (-1)(-2)\,a^3(ax + b)^{-3}$, etc. In general

$$\mathbf{D}^n \log(ax + b) = \frac{(-1)^{n-1}(n-1)!\,a^n}{(ax + b)^n}.$$

(5) If $y = \sin(ax + b)$,

$$y_1 = a\cos(ax + b) = a\sin(ax + b + \tfrac{1}{2}\pi),$$
$$y_2 = a^2\cos(ax + b + \tfrac{1}{2}\pi)$$
$$= a^2\sin(ax + b + \pi),$$
$$y_3 = a^3\sin(ax + b + 3\pi/2)\text{; etc.}$$

In general, $\mathbf{D}^n \sin(ax + b) = a^n \sin(ax + b + \tfrac{1}{2}n\pi).$

(6) Similarly

$$\mathbf{D}^n \cos(ax + b) = a^n \cos(ax + b + \tfrac{1}{2}n\pi).$$

Corollaries. Putting $a = 1$ and $b = 0$, we have

$$D^n \sin x = \sin(x + \tfrac{1}{2}n\pi),$$

and $$D^n \cos x = \cos(x + \tfrac{1}{2}n\pi).$$

(7) If $y = e^{ax}\sin(bx + c)$, then

$$y_1 = e^{ax}\,b\cos(bx + c) + ae^{ax}\sin(bx + c).$$

Putting $a = r\cos\varphi$ and $b = r\sin\varphi$, we have

$$y_1 = re^{ax}\sin(bx + c + \varphi).$$

Similarly, $y_2 = r^2e^{ax}\sin(bx + c + 2\varphi)$; etc.

In general

$$\mathbf{D}^n\{e^{ax}\sin(bx + c)\} = r^n\,e^{ax}\sin(bx + c + n\varphi),$$

where $r = (a^2 + b^2)^{1/2}$, and $\varphi = \tan^{-1}(b/a)$.

(8) Similarly

$$\mathbf{D}^n\{\mathbf{e}^{ax}\cos(\mathbf{bx}+\mathbf{c})\} = \mathbf{r}^n\,\mathbf{e}^{ax}\cos(\mathbf{bx}+\mathbf{c}+\mathbf{n}\varphi),$$

where r and φ have the same meaning as before.

5·3. Decomposition into a sum. Before applying the above standard results to particular functions, it is often necessary to break up the given function into a sum of suitable functions. Some methods for this are dealt with below.

5·31. Partial Fractions. For finding the nth differential coefficient of a fraction whose numerator and denominator are both rational integral algebraic functions, the given fraction must be decomposed into its partial fractions.

Ex. Find the nth differential coefficient of $\dfrac{x^3}{(x-1)(x-2)}$.

We know by algebra that

$$\frac{x^3}{(x-1)(x-2)} = x + 3 + \frac{7x-6}{(x-1)(x-2)}$$

$$= x + 3 - \frac{1}{x-1} + \frac{8}{x-2}.$$

Therefore, if $n > 1$, $\dfrac{d^n}{dx^n}\dfrac{x^3}{(x-1)(x-2)}$

$$= (-1)^{n+1}(n!)\left\{\frac{1}{(x-1)^{n+1}} - \frac{8}{(x-2)^{n+1}}\right\}.$$

5·32. Use of De M[illegible] n. Even wh[illegible]
we cannot break up th[illegible] n
fraction into real lin[illegible]
used after resolving
real or imaginary

Ex. Find th[illegible]

If $y = \tan^{-1}$ [illegible]

Now $\dfrac{1}{a^2+x}$ [illegible]

Therefore, $y_n = \dfrac{a(-1)^{n-1}(n-1)!}{2ia}\left\{\dfrac{1}{(x-ia)^n} - \dfrac{1}{(x+ia)^n}\right\}$.

Put $x = r\cos\phi$, $a = r\sin\phi$; then

$$\begin{aligned} y_n &= \tfrac{1}{2}(-1)^n (n-1)!\, ir^{-n}\{(\cos\phi - i\sin\phi)^{-n} \\ &\qquad\qquad - (\cos\phi + i\sin\phi)^{-n}\} \\ &= \tfrac{1}{2}(-1)^n (n-1)!\, ir^{-n}\{(\cos n\phi + i\sin n\phi) \\ &\qquad\qquad - (\cos n\phi - i\sin n\phi)\} \\ &= (-1)^{n+1}(n-1)!\, r^{-n}\sin n\phi. \end{aligned}$$

But $$r^{-n} = a^{-n}\sin^n\phi.$$

Hence $D^n\tan^{-1}(x/a) = (-1)^{n-1}(n-1)!\, a^{-n}\sin^n\phi\sin n\phi$,

where $$\phi = \tan^{-1}(a/x).$$

5·33. Trigonometrical transformation. It is possible to break up products of powers of sines and cosines into a sum by Trigonometry.

Ex. 1. Find the nth differential coefficient of $\sin^5 x\cos^3 x$.

Let $\cos x + i\sin x = z$; then $\cos x - i\sin x = z^{-1}$.

Therefore $2\cos x = z + z^{-1}$, $2i\sin x = z - z^{-1}$.

Also by De Moivre's Theorem, $2\cos px = z^p + z^{-p}$;

$$2i\sin px = z^p - z^{-p}.$$

Therefore $2^5\, 2^3\, i^5 \sin^5 x\cos^3 x = (z - z^{-1})^5(z + z^{-1})^3$

$$\begin{aligned} &= (z^8 - z^{-8}) - 2(z^6 - z^{-6}) - 2(z^4 - z^{-4}) + 6(z^2 - z^{-2}) \\ &= 2i\sin 8x - 4i\sin 6x - 4i\sin 4x + 12i\sin 2x. \end{aligned}$$

Therefore

$$\begin{aligned} D^n(\sin^5 x\cos^3 x) = 2^{-7}\{&8^n\sin(8x + \tfrac{1}{2}n\pi) - 2\cdot 6^n\sin(6x + \tfrac{1}{2}n\pi) \\ &- 2\cdot 4^n\sin(4x + \tfrac{1}{2}n\pi) + 6\cdot 2^n\sin(2x + \tfrac{1}{2}n\pi)\}. \end{aligned}$$

Ex. 2. Find the ... ential coefficient of $\sin x\cos 3x$.

Here ... $(\sin 4x - \sin 2x)$.

D^n ...

... $\sin(2x + \tfrac{1}{2}n\pi)\}$.

... of x^4e^{5x}, ... x^2,

... $_2 + m^2 y = 0$.

... $+ b^2)y = 0$.

[*Calcutta*, 1936]

Find the nth differential coefficients of

4. e^{ax+b}.
5. $(ax+b)^{p/q}$.
6. $\log\{(ax+b)(cx+d)\}$.
7. $\sin 2x \sin 3x$.
8. $\cos x \cos 2x \cos 3x$. [*M.T.*, '30]
9. $\cos^4 x$. [*Punjab*, 1936]
10. $\cos^2 x \sin^3 x$. [*All.*, '23]
11. $e^{ax} \cos^3 bx$.

$\sin 3x - 3\sin x - 4\sin^3 x$

12. $e^{ax} \sin bx \cos cx$.
13. $(a^2 - x^2)^{-1}$.
14. $\dfrac{x^4}{(x-1)(x-2)}$. [*M.T.*, '33]
15. $\dfrac{1}{1-5x+6x^2}$.
16. $x/(x-a)(x-b)(x-c)$.
17. $(ax+b)/(cx+d)$.
18. $x^2/(1-x^4)$. [*Andhra*, '36]
19. $\tan^{-1}(1+x)/(1-x)$.

20. Find the nth differential coefficient of $e^{ax} \sin bx$ and deduce the nth differential coefficient of $\sin x \sin bx$. [*Allahabad*, 1923]

5·4. Leibnitz's Theorem.* This theorem is useful for finding the nth differential coefficient of a product. It is as follows:

If u and v be any two functions of x,

$$\mathbf{D}^n(\mathbf{uv}) = (\mathbf{D}^n\mathbf{u}).\mathbf{v} + {}^n\mathbf{C}_1\, \mathbf{D}^{n-1}\mathbf{u}.\mathbf{D}\mathbf{v} + {}^n\mathbf{C}_2\, \mathbf{D}^{n-2}\mathbf{u}.\mathbf{D}^2\mathbf{v} + \dots + {}^n\mathbf{C}_r\, \mathbf{D}^{n-r}\mathbf{u}.\mathbf{D}^r\mathbf{v} + \dots + \mathbf{u}.\mathbf{D}^n\mathbf{v}.$$

We shall prove this theorem by mathematical induction. Assume that the above result is true for a particular value of n. Then, differentiating with respect to x, we have

$$\begin{aligned} D^{n+1}(uv) = {} & \{(D^{n+1}u).v + D^n u.Dv\} \\ & + {}^nC_1\{D^n u.\, Dv + D^{n-1}u.\, D^2 v\} + \dots \\ & + {}^nC_r\{D^{n-r+1}u.\, D^r v + D^{n-r}u.\, D^{r+1}v\} + \dots \\ & + \{Du.\, D^n v + u.D^{n+1}v\}. \end{aligned}$$

Re-arranging, we have

$$\begin{aligned} D^{n+1}(uv) = {} & (D^{n+1}u).v + (1 + {}^nC_1)(D^n u.\, Dv) + \dots \\ & + ({}^nC_r + {}^nC_{r+1})(D^{n-r}u.\, D^{r+1}v) + \dots + u.D^{n+1}v. \end{aligned}$$

*Named after the German mathematician Gottfried Wilhelm Leibnitz (1646-1716), who was an independent inventor of the calculus, and to whom our notation for differentiation is due. (See the Historical Note at the end of this book.)

But $\qquad {}^nC_r + {}^nC_{r+1} = {}^{n+1}C_{r+1}.$

Hence $D^{n+1}(u\,v) = (D^{n+1}u).v + {}^{n+1}C_1\, D^nu.Dv + \ldots$
$$+ {}^{n+1}C_{r+1}D^{n-r}u.D^{r+1}v + \ldots + u.D^{n+1}v.$$

Therefore, if the theorem is true for any value of n, it is true for the next value of n.

But it is easy to see that it holds for $n = 2$, for
$$D(u\,v) = (Du).\,v + u.\,Dv,$$
therefore $\qquad D^2(u\,v) = (D^2u).v + 2\,Du.Dv + u.D^2v.$

Hence the theorem must be true when $n = 3$, and so when $n = 4$; and so on. Thus it must be true for every positive integral value of n.

Ex. 1. Find the nth differential coefficient of $x^3\,e^{ax}$.

Choosing, for the sake of convenience, x^3 to be the second function, we have at once $D^n(x^3.e^{ax})$
$$= a^n\,e^{ax}.\,x^3 + {}^nC_1\,a^{n-1}e^{ax}.\,3x^2 + {}^nC_2\,a^{n-2}\,e^{ax}.\,6x + {}^nC_3\,a^{n-3}\,e^{ax}.\,6.$$

This can now be simplified.

Ex. 2. Differentiate n times the equation
$$(1 - x^2)\frac{d^2y}{dx^2} - x\,\frac{dy}{dx} + a^2y = 0. \qquad [\textit{Allahabad}, 1926]$$

Here
$$D^n\,\{(1 - x^2)y_2\} = (1 - x^2)y_{n+2} + n.\,(-2x)y_{n+1} + \{n(n-1)/2!\}(-2)y_n,$$
$$D^n\,(-xy_1) = \qquad -x\,y_{n+1} \qquad - ny_n,$$
$$D^n\,(a^2\,y) = \qquad + a^2y_n.$$

Adding, $\qquad 0 = (1 - x^2)\,y_{n+2} - (2n + 1)\,xy_{n+1} - (n^2 - a^2)\,y_n,$

i.e., $\qquad (1 - x^2)\dfrac{d^{n+2}y}{dx^{n+2}} - (2n + 1)\,x\,\dfrac{d^{n+1}y}{dx^{n+1}} - (n^2 - a^2)\,\dfrac{d^ny}{dx^n} = 0.$

Examples

1. Find the 4th differential coefficients of x^2e^{2x}, $x^3 \log x$, $x^{\frac{1}{2}}\sin 3x$, $(\log x)/(x + a)$, $x\,e^{ax}\sin bx$.

Find the nth differential coefficients of

2. $x^2\,e^{ax}$. $\qquad$ 3. $e^x\,(ax + b)^3$.

4. $x^3 \cos x$. [*M. T.*, '25] 5. $x^2 (ax + b)^m$.

6. $x^3 \log x$. 7. $\sin x \log (ax + b)$.

8. $e^x \log x$. [*M. T.*, '25] 9. $x^2 \tan^{-1} x$.

10. If $y = a \cos (\log x) + b \sin (\log x)$, show that

$$x^2 y_2 + xy_1 + y = 0,$$

and $$x^2 y_{n+2} + (2n + 1)\, xy_{n+1} + (n^2 + 1)\, y_n = 0. \qquad \text{[Benares, 1933]}$$

11. If $y = A\, e^{-kt} \cos (pt + e)$, show that

$$\frac{d^2y}{dt^2} + 2k \frac{dy}{dt} + n^2 y = 0,$$

where $$n^2 = p^2 + k^2. \qquad \text{[Madras, 1936]}$$

12. If $y = x^2 e^x$, show that

$$\frac{d^n y}{dx^n} = \tfrac{1}{2} n(n - 1) \frac{d^2 y}{dx^2} - n(n - 2) \frac{dy}{dx} + \tfrac{1}{2}(n - 1)(n - 2)y.$$

[*Nagpur*, 1934]

§ 41. ***n*th differential coefficient for special values of *x*.** Sometimes, although we may not be able to find the *n*th differential coefficient in a compact form for the general value of x, we can find the *n*th differential coefficient for some special value of x. The procedure will be clear from the following example.

Ex. Find $(y_n)_0$ when $y = \sin (a \sin^{-1} x)$.

Here $$y_1 = \cos (a \sin^{-1} x) \cdot \frac{a}{\sqrt{(1 - x^2)}}, \qquad \ldots \ldots (1)$$

or $$(1 - x^2) y_1^2 = a^2 \cos^2 (a \sin^{-1} x),$$

i.e., $$(1 - x^2) y_1^2 = a^2 (1 - y^2). \qquad \ldots \ldots (2)$$

By differentiation, $(1 - x^2) \cdot 2y_1 y_2 - 2xy_1^2 = -2yy_1 a^2$.

Dividing out by $2y_1$,

$$(1 - x^2)\, y_2 - xy_1 + a^2 y = 0. \qquad \ldots \ldots (3)$$

Differentiating this n times by Leibnitz's Theorem (see Ex. 2 of the previous article):

$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - (n^2 - a^2) y_n = 0. \qquad . (4)$$

Putting $x = 0$, this gives

$$(y_{n+2})_0 = (n^2 - a^2)\, (y_n)_0. \qquad \ldots \ldots (5)$$

But by (1) and (3), $(y_1)_0 = a$, and $(y_2)_0 = 0$.

Hence, by (5), $y_2, y_4, y_6, \ldots$ are all zero at $x = 0$, and if n is odd

$$(y_n)_0 = \{(n - 2)^2 - a^2\}\, (y_{n-2})_0$$
$$= \{(n - 2)^2 - a^2\}\, \{(n - 4)^2 - a^2\}\, (y_{n-4})_0 = \text{etc.}$$

Thus $(y_n)_0 = \{(n - 2)^2 - a^2\}\, \{(n - 4)^2 - a^2\}\, \{(n - 6)^2 - a^2\} \ldots \{1 - a^2\} a$.

EXAMPLES

1. If $y = [\log \{x + \sqrt{(1 + x^2)}\}]^2$, show that

$$(y_{n+2})_0 = -n^2(y_n)_0 ;$$

hence find $(y_n)_0$.

2. If $y = \sin^{-1} x$, find $(y_n)_0$.

3. If $u = \tan^{-1} x$, prove that

$$(1 + x^2)\frac{d^2u}{dx^2} + 2x\frac{du}{dx} = 0,$$

and hence determine the values of all the derivatives of u with respect to x, when $x = 0$. [*Math. Tripos*, 1931]

EXAMPLES ON CHAPTER V

1. If $p^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$, prove that

$$p + \frac{d^2p}{d\theta^2} = \frac{a^2 b^2}{p^3}.$$ [*Agra*, 1929]

2. Prove that the value of the nth differential coefficient of $x^3/(x^2 - 1)$ for $x = 0$ is zero if n is even, and is $-n!$ if n is odd and greater than 1. [*Math. Tripos*, 1935]

3. Find the nth differential coefficient of $\tan^{-1}\{2x/(1 - x^2)\}$. [*Calcutta*, 1938]

4. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, show that

$$D^2y = \Delta\,(hx + by + f)^{-3},$$

where $$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2.$$ [*Punjab*, 1930]

5. Find the nth differential coefficient of $x^{n-1} \log x$. [*Dacca*, '35]

6. If $y^{1/m} + y^{-1/m} = 2x$, prove that

$$(x^2 - 1)\,y_{n+2} + (2n + 1)\,xy_{n+1} + (n^2 - m^2)\,y_n = 0,$$

where y_n denotes the nth derivative of y. [*Bombay*, 1937]

7. Prove that the nth differential coefficient of $x^n(1 - x)^n$ is equal to

$$n!\,(1 - x)^n\left\{1 - \frac{n^2}{1^2}\frac{x}{1 - x} + \frac{n^2(n - 1)^2}{1^2.\,2^2}\frac{x^2}{(1 - x)^2} + \dots\right\}.$$

8. If $y = x(a^2 + x^2)^{-1}$, prove that

$$y_n = (-1)^n\, n!\, a^{-n-1} \sin^{n+1}\phi \cos(n + 1)\phi,$$

where $$\phi = \tan^{-1}(a/x).$$

9. Show that if $u = \sin nx + \cos nx$, then

$$u_r = n^r \{1 + (-1)^r \sin 2nx\}^{1/2},$$

where u_r denotes the rth differential coefficient of u with respect to x. [*Lucknow*, 1934]

10. Prove that the value when $x = 0$ of $D^n (\tan^{-1} x)$ is 0, $(n-1)!$ or $-(n-1)!$ according as n is of the form $2p$, $4p+1$, or $4p+3$ respectively. Ex 3 p. 72. [*I. C. S.*, 1929]

11. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\sin x}{x}\right) = \left\{P \sin (x + \tfrac{1}{2}n\pi) + Q \cos (x + \tfrac{1}{2}n\pi)\right\} / x^{n+1},$$

where

$$P = x^n - n(n-1)\, x^{n-2} + n(n-1)(n-2)(n-3)\, x^{n-4} - \ldots,$$

and $Q = n\, x^{n-1} - n(n-1)(n-2)\, x^{n-3} + \ldots$

12. If $y = e^{a \sin^{-1} x}$, prove that

$$(1 - x^2)\, y_{n+2} - (2n+1)\, x y_{n+1} - (n^2 + a^2) y_n = 0,$$

and $\lim_{x \to 0} (y_{n+2}/y_n) = n^2 + a^2$. [*Allahabad*, 1927]

13. If $y = (\sin^{-1} x)^2$, prove that

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 2 = 0.$$

Differentiate the above equation n times with respect to x. [*Agra*, 1937]

14. Find the nth differential coefficient, at $x = 0$, of $e^{m \cos^{-1} x}$.

15. If $y = (x^2 - 1)^n$, prove that

$$(x^2 - 1) y_{n+2} + 2x y_{n+1} - n(n+1) y_n = 0.$$

Hence, if $P_n = \dfrac{d^n}{dx^n} (x^2 - 1)^n$, show that

$$\frac{d}{dx}\left\{(1 - x^2)\frac{dP_n}{dx}\right\} + n(n+1) P_n = 0.$$

[*Agra*, 1933]

16. If $\cos^{-1} (y/b) = \log (x/n)^n$, prove that

$$x^2 y_{n+2} + (2n+1)\, x y_{n+1} + 2n^2 y^n = 0.$$

[*Nagpur*, 1935]

17. If $y = (x + \sqrt{1 + x^2})^m$, find $(y_n)_{x=0}$.

18. If $Y = sX$ and $Z = tX$, all the variables being functions of x, prove that

$$\begin{vmatrix} X & Y & Z \\ X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = X^3 \begin{vmatrix} s_1 & t_1 \\ s_2 & t_2 \end{vmatrix}.$$

[*Benares*, 1935]

19. Show that the $2n$th differential coefficient of $\cos^{2m} x$ is
$(-)^n\, 2^{-2m+1} [(2m)^{2n}\, C_0 \cos 2mx$

$$+ (2m-2)^{2n}\, C_1 \cos (2m-2)x + \ldots + 2^{2n}\, C_{m-1} \cos 2x],$$

where C_r is the coefficient of x^r in the expansion of $(1+x)^{2m}$ in powers of x. [*Math. Tripos*, 1928]

20. By forming in two different ways the nth derivative of x^{2n}, prove that

$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots = \frac{(2n)!}{(n!)^2}.$$

[First find the nth derivative of the product of x^n and x^n; next that of x^{2n}.] → & divide the two results

21. Prove by induction or otherwise that if D means $\cos^2\theta \dfrac{d}{d\theta}$, then

$$D^n(\tfrac{1}{2}\sin 2\theta) = (-1)^n.\, n!\, (\cos\theta)^{n+1} \cos\{(n+1)(\tfrac{1}{2}\pi - \theta)\}.$$

[*Lucknow*, 1930]

22. If $y = \dfrac{b+cx}{a+2bx+cx^2}$, where $ac > b^2$; prove that → it is of the form

$$y_n = (-1)^n (n!) \left(\frac{c}{a+2bx+cx^2}\right)^{(n+1)/2}$$

$\frac{z}{z^2+a^2}$ then $z =$

$$\times \cos\left\{(n+1)\tan^{-1}\frac{\sqrt{(ac-b^2)}}{b+cx}\right\}.$$

[*Patna*, 1933]

23. If $x + y = 1$, prove that $\therefore \frac{dy}{dx} = -1.$

$$\frac{d^n}{dx^n}(x^n y^n) = n!\,\{y^n - ({}^nC_1)^2 y^{n-1} x + ({}^nC_2)^2 y^{n-2} x^2 + \ldots + (-)^n x^n\}.$$

[*Madras*, 1935]

CHAPTER VI

EXPANSION OF FUNCTIONS

6·1. Infinite series. The ordinary processes of addition, subtraction, multiplication, division, rearrangement of terms, raising to a given power, taking limits, differentiation, etc., though applicable to the sum of a finite number of terms, may break down for infinite series. The expansions in the form of infinite series obtained by the methods given below are therefore to be regarded merely as formal expansions, which may not be true in exceptional cases. That is why the expansion in the form of the sum of a finite number of terms obtained in § 6·9 is generally preferred.

6·2. Maclaurin's Theorem. Let $f(x)$ be a function of x which can be expanded in powers of x and let the expansion be differentiable term by term any number of times.

Suppose $f(x) = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + \dots$
Then by successive differentiation we have

$$f'(x) = A_1 + 2A_2 x + 3A_3 x^2 + 4A_4 x^3 + \dots$$
$$f''(x) = 2.1A_2 + 3.2.A_3 x + 4.3A_4 x^2 + \dots$$
$$f'''(x) = 3.2.1A_3 + 4.3.2A_4 x + \dots\text{; etc. etc.}$$

Putting $x = 0$ in each of these, we get

$$f(0) = A_0,\ f'(0) = A_1,\ f''(0) = 2!A_2,\ f'''(0) = 3!A_3, \dots$$

Hence* $$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots}$$

*This result is generally known as Maclaurin's Theorem after Colin Maclaurin (1698-1746), professor of mathematics at the University of Edinburgh. It was discovered by Stirling (1717) and published by Maclaurin (1742) in his *Fluxions*.

Ex. Expand $\sin x$ by Maclaurin's theorem.

Let
$$
\begin{aligned}
f(x) &= \sin x, & \text{then} \quad f(0) &= 0,\\
f'(x) &= \cos x, & f'(0) &= 1,\\
f''(x) &= -\sin x, & f''(0) &= 0,\\
f'''(x) &= -\cos x, & f'''(0) &= -1,\\
&\text{etc.} & &\text{etc.}\\
f^{(n)}(x) &= \sin(x + \tfrac{1}{2}n\pi), & f^{(n)}(0) &= \sin \tfrac{1}{2}n\pi\\
& & &= 0 \text{ if } n = 2m,\\
& & \text{and } &= (-1)^m \text{ if } n = 2m+1.
\end{aligned}
$$

Hence
$$
\begin{aligned}
\sin x &= 0 + x.1 + 0 + \frac{x^3}{3!}(-1) + 0 + \dots\\
&\qquad + 0 + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \dots\\
&= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^m \frac{x^{2m+1}}{(2m+1)!} + \dots
\end{aligned}
$$

NOTE. When the nth differential coefficient of the function cannot be found, the nth term of the expansion cannot be ascertained. It is possible, however, that the nth differential coefficient be known for $x = 0$ (see § 5·41).

EXAMPLES

Expand the following functions by Maclaurin's theorem:

1. $\cos x$.
2. $(1 + x)^m$.
3. $\log(1 + x)$.
4. e^x.
5. $\sin^{-1} x$.
6. $e^{\sin x}$. [*Nag.*, '32]
7. $\tan x$. [*Andhra*, 1937]
8. $\tan^{-1} x$. [*Calcutta*, 1936]
9. $e^{x \cos x}$. [*P.C.S.*, '31]
10. $e^{a \cos^{-1} x}$.
11. $\sec x$. [*Punjab*, 1932]
12. $e^x \sec x$. [*Ben.*, '32]
13. $e^x \log(1 + x)$. [*Dacca*, '36]
14. $\sin x \sinh x$.
15. $\log(1 + \sin^2 x)$. [*Agra*, '33]
16. Show that

$$e^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{2^2 x^4}{4!} - \frac{2^2 x^5}{5!} + \frac{2^3 x^7}{7!} + \dots$$

[*Allahabad*, 1925]

17. Apply Maclaurin's theorem to prove that

$$\log \sec x = \tfrac{1}{2}x^2 + \tfrac{1}{12}x^4 + \tfrac{1}{45}x^6 + \dots \qquad \text{[\textit{Agra}, 1929]}$$

18. Use Maclaurin's theorem to find the expansion in ascending powers of x of $\log_e (1 + e^x)$ to the term containing x^4. [*Nag.*, 1935]

19. Show that the first five terms in the power series for $\log (1 + \sin x)$ are

$$x - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 - \tfrac{1}{12}x^4 + \tfrac{1}{24}x^5. \qquad [\textit{Patna}, 1937]$$

*20. Apply Maclaurin's theorem to obtain the expansion of the function $e^{ax} \cos bx$ in an infinite series of powers of x, giving the general term. [*Agra*, 1934]

6·3. Taylor's Theorem.

Let $f(a + h)$ be a function of h which can be expanded in powers of h, and let the expansion be differentiable any number of times with respect to h.

Suppose $f(a + h) = A_0 + A_1 h + A_2 h^2 + A_3 h^3 + \ldots$

By successive differentiations with respect to h, we have

$$f'(a + h) = A_1 + 2A_2 h + 3A_3 h^2 + \ldots,$$

for

$$\frac{d}{dh} f(a + h) = \frac{d}{dt} f(t) \cdot \frac{dt}{dh}, \text{ where } t = a + h$$

$$= f'(t) \cdot 1 = f'(a + h);$$

$$f''(a + h) = 2 \cdot 1 \, A_2 + 3 \cdot 2 \, A_3 h + \ldots,$$

$$f'''(a + h) = 3 \cdot 2 \cdot 1 \, A_3 + \ldots,$$

etc.

Putting $h = 0$ in each of these, we have

$$f(a) = A_0,\ f'(a) = A_1,\ f''(a) = 2!\, A_2,$$

$$f'''(a) = 3!\, A_3, \ldots$$

Hence

$$\mathbf{f(a + h) = f(a) + h\,f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \ldots + \frac{h^n}{n!} f^{(n)}(a) + \ldots}$$

This is known as Taylor's Theorem.*

*Named after the English mathematician and philosopher, Brook Taylor (1685-1731). The formal result was first published by him in his *Methodus Incrementorum* (1715). A real proof was first given a century later by Cauchy.

If we put $a = 0$, and $b = x$, we get the particular case known as Maclaurin's Theorem. For this reason the latter theorem also is very often referred to by the name of Taylor's Theorem.

Taylor's theorem is generally quoted as

$$\mathbf{f(x + h) = f(x) + h\,f'(x) + \frac{h^2}{2!}\,f''(x) + \frac{h^3}{3!}\,f'''(x) + \ldots + \frac{h^n}{n!}\,f^{(n)}(x) + \ldots,}$$

which is obtained at once from the above on replacing a by x. A more useful form is obtained on replacing b by $(x - a)$. Thus

$$\mathbf{f(x) = f(a) + (x - a)\,f'(a) + \frac{(x-a)^2}{2!}\,f''(a) + \frac{(x-a)^3}{3!}\,f'''(a) + \ldots + \frac{(x-a)^n}{n!}\,f^{(n)}(a) + \ldots,}$$

which is an expansion in powers of $(x - a)$.

Ex. 1. Expand $\log \sin (x + h)$ in powers of h by Taylor's theorem.

Let $\quad f(x) = \log \sin x.$

Then $\quad f'(x) = \cot x,$

$$f''(x) = -\operatorname{cosec}^2 x,$$

$$f'''(x) = +\,2 \operatorname{cosec} x \operatorname{cosec} x \cot x, \text{ etc.}$$

Hence $\log \sin (x + h) = \log \sin x + h \cot x - \frac{1}{2} h^2 \operatorname{cosec}^2 x + \frac{1}{3} h^3 \operatorname{cosec}^2 x \cot x + \ldots$

Ex. 2. Expand $\log \sin x$ in powers of $x - 2$.

We can write $\log \sin x$ as $\log \sin (2 + x - 2)$. Hence, replacing x by 2 and h by $x - 2$ in the above, we have

$$\log \sin x = \log \sin 2 + (x - 2) \cot 2 - \tfrac{1}{2}(x - 2)^2 \operatorname{cosec}^2 2 + \tfrac{1}{3}(x - 2)^3 \operatorname{cosec}^2 2 \cot 2 + \ldots$$

Examples

1. Show that $\log (x + h) = \log h + \dfrac{x}{h} - \dfrac{x^2}{2h^2} + \dfrac{x^3}{3h^3} + \ldots$

2. Expand $\sin^{-1} (x + h)$ in powers of x as far as the term in

Expand the following in powers of the quantity indicated:

3. e^{2x} in powers of $(x + 1)$.

4. $\tan^{-1} x$ in powers of $(x - \frac{1}{4}\pi)$.

5. $\sin (\frac{1}{4}\pi + \theta)$ in powers of θ. [*Annamalai*, 1936]

6. $2x^3 + 7x^2 + x - 1$ in powers of $x - 2$.

7. Use Taylor's theorem to prove that

$$\tan^{-1}(x+h) - \tan^{-1}x = h\sin z \frac{\sin z}{1} - (h\sin z)^2\frac{\sin 2z}{2} + (h\sin z)^3\frac{\sin 3z}{3} \ldots,$$

where $z = \cot^{-1} x$. [*Agra*, 1932]

6·4. Approximate calculations. It is often necessary to know what is the increment in any quantity when the variable on which it depends is changed by a small amount. This we can find at once by Taylor's theorem. Thus if $y = f(x)$,

$$y + \delta y = f(x + \delta x) = f(x) + \delta x f'(x) + (1/2\,!)(\delta x)^2 f''(x) + \ldots$$

Hence $$\delta y = \frac{dy}{dx}\delta x + \frac{1}{2!}\frac{d^2y}{dx^2}(\delta x)^2 + \ldots$$

If the second and higher powers of δx can be neglected, we have the formula of § 4·4.

6·41. Orders of small quantities. If x is small in comparison with unity, and we take x to be a small quantity of order 1, then x^n is called a small quantity of order n. Moreover, if A is nearer to x^n than it is to x^{n-1} or x^{n+1}, A also will be called a small quantity of order n. How small x should be in order that it may be called a small quantity of order 1 is perfectly arbitrary. In approximate results the terms of order higher than a certain one are neglected, the number of the terms retained depending on the accuracy desired and the rapidity with which the terms diminish.

6·5. Expansion by Algebraic and Trigonometrical Methods. When we want only the first few terms of an expansion, it is very often more convenient to use the Binomial, Exponential or the Logarithmic theorems, or the well-known expansion of $\sin x$ or $\cos x$, in conjunction with algebraic or trigonometrical methods.

But *care should be taken not to inadvertently omit any term of an order lower than, or the same as, that of the highest order term retained.*

Ex. Expand $e^{\sin x}$ as far as the term containing x^4.

We have $e^{\sin x} = e^{(x - x^3/6 + \ldots)}$

$$= 1 + (x - \tfrac{1}{6}x^3 + \ldots) + \tfrac{1}{2}(x - \tfrac{1}{6}x^3 + \ldots)^2 + \tfrac{1}{6}(x - \tfrac{1}{6}x^3 + \ldots)^3 + \tfrac{1}{24}(x - \ldots)^4 + \ldots$$

$$= 1 + x - \tfrac{1}{6}x^3 + \dots + \tfrac{1}{2}(x^2 - \tfrac{1}{3}x^4 + \dots)$$
$$+ \tfrac{1}{6}(x^3 + \dots) + \tfrac{1}{24}(x^4 + \dots) + \dots$$
$$= 1 + x + \tfrac{1}{2}x^2 - \tfrac{1}{8}x^4 + \dots$$

6·6. By Differentiation of a known series. Occasionally it is possible to derive an expansion by differentiating a known series.

Ex Expand $\sec^2 x$

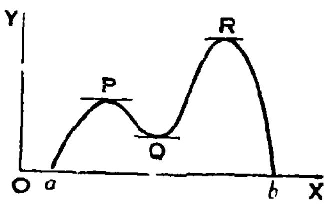
We know that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

Hence, by differentiation, $\sec^2 x = 1 + x^2 + \frac{2}{3}x^4 + \dots$

6·7. Rolle's Theorem*. *If $f(x)$ is zero when $x = a$ and also when $x = b$, and if $f(x)$ possesses a differential coefficient $f'(x)$ for every value of x in the domain (a, b), then $f'(x)$ will be zero for at least one value of x between a and b.*

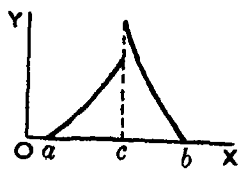
Throughout this chapter the numbers a and b are supposed to be included in the domain (a, b)

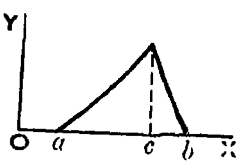
The truth of this theorem is obvious from geometrical intuition; for, if $f(a) = 0$ and $f(b) = 0$, there must be at least one point like P where $f'(x)$ is zero, i.e., where the tangent is parallel to the x-axis. There might be three such points (as in the figure), or even more.



If $f(x)$ is discontinuous at any point between a and b, or if $f'(x)$ does not exist at some point, the theorem may not be true

Thus, in the figures given below, $f'(x)$ does not exist at $x = c$. In these cases there is no point between a and b where the tangent to the curve is parallel to the x-axis





*Called after the French mathematician Michel Rolle (1652-1719), who first gave it in his book *Methode pour résoudre les egalites* (1691).

The analytical proof of Rolle's theorem is beyond the scope of this book.

6·8. The Mean Value Theorem. *If $f(x)$ possesses a differential coefficient for every value of x in the domain (a, b), then there must be a value (say x_1) of x, between a and b, such that*

$$\mathbf{f(b) - f(a) = (b - a)\, f'(x_1).}$$

Consider the function $F(x)$ defined by

$$F(x) = f(b) - f(x) - \frac{f(b) - f(a)}{b - a}(b - x).$$

Putting $x = a$ and $x = b$ by turns we see that $F(a) = 0$, and $F(b) = 0$. Also, by supposition $f'(x)$ exists for every value of x from a to b, so $F'(x)$ also exists, its value being given by

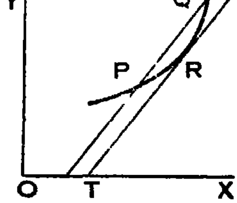
$$F'(x) = -f'(x) + \frac{f(b) - f(a)}{b - a}. \quad . \quad . \quad (1)$$

Therefore, applying Rolle's theorem, $F'(x)$ vanishes for at least one value (say x_1) of x between a and b. Substituting this value of x in (1), we have

$$0 = -f'(x_1) + \frac{f(b) - f(a)}{b - a},$$

which proves the proposition.

6·81. Geometrical Meaning of the Mean Value Theorem. Let P and Q be two points on the curve $y = f(x)$. Further, let the abscissæ of P and Q be a and b respectively. Then if PQ makes with the x axis an angle ψ, we have, by geometry



$$\tan\psi = \frac{f(b) - f(a)}{b - a}.$$

Also, if the tangent to the curve at R (x_1, y_1) makes an angle ψ_1 with the x axis, we have by § 4·6,

$$\tan\psi_1 = f'(x_1).$$

Hence the theorem of the mean value merely asserts that there is some point between P and Q where the tangent is parallel to PQ, provided the curve has a tangent at every point between P and Q. This is geometrically obvious.

6·82. Another form of the Mean Value Theorem. If we put $b - a = h$, then $a + \theta h$ is equal to a if $\theta = 0$, and is equal to b if $\theta = 1$.

Hence $a + \theta h$, where $0 < \theta < 1$, means some value between a and b. So the above theorem can be written as

$$f(a + h) - f(a) = hf'(a + \theta h),$$

or $$\mathbf{f(a + h) = f(a) + h\,f'(a + \theta h)}, \text{ where } 0 < \theta < 1.$$

6·9. Taylor's Theorem. Finite Form. A more general theorem than the one of Mean Value is Taylor's theorem. The method of proof is similar to that of § 6·8. The theorem is as follows:

If $f(x)$ possesses differential coefficients of the first n orders for every value of x in the domain (a, b), then

$$\mathbf{f(b) = f(a) + (b - a)\,f'(a) + \frac{(b - a)^2}{2!}\,f''(a) + \dots}$$
$$\mathbf{+ \frac{(b - a)^r}{r!}\,f^{(r)}(a) + \dots + \frac{(b - a)^{n-1}}{(n - 1)!}\,f^{(n-1)}(a)}$$
$$\mathbf{+ \frac{(b - a)^n}{n!}\,f^{(n)}\{a + \theta(b - a)\}},$$

for some value of θ such that $0 < \theta < 1$.

Consider the function $F(x)$ given by

$$F(x) = f(b) - f(x) - (b - x)f'(x)$$
$$- \frac{(b - x)^2}{2!}f''(x) - \dots - \frac{(b - x)^r}{r!}f^{(r)}(x) - \dots$$
$$- \frac{(b - x)^{n-1}}{(n - 1)!}f^{(n-1)}(x) - \frac{(b - x)^n}{n!}Q,$$

where Q is a constant given by the equation $F(a) = 0$, i.e., by

$$0 = f(b) - f(a) - (b - a)f'(a) - \dots$$
$$- \{1/(n - 1)!\}\,(b - a)^{n-1}f^{(n-1)}(a)$$
$$- (1/n!)\,(b - a)^n Q. \quad . \quad . \quad . \quad (1)$$

Substituting b for x, we see that $F(b)$ also is zero. Now $F(x)$ consists of a finite number of terms the differential coefficients of all of which exist, and so $F'(x)$ exists for every value of x from a to b. Hence we can apply Rolle's

theorem, which shows that $F'(x)$ vanishes for at least one value (say x_1) of x between a and b.

Now differentiating $F(x)$ we get

$$F'(x) = -f'(x) + f'(x) - (b-x)f''(x)$$
$$+ (b-x)f''(x) - \frac{(b-x)^2}{2!}f'''(x) + \dots$$
$$- \frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x) + \frac{(b-x)^{n-1}}{(n-1)!}Q,$$

i.e.,
$$F'(x) = \frac{(b-x)^{n-1}}{(n-1)!}\{Q - f^{(n)}(x)\}.$$

Hence the equation $F'(x_1) = 0$ becomes

$$\frac{(b-x_1)^{n-1}}{(n-1)!}\{Q - f^{(n)}(x_1)\} = 0, \text{ i.e., } Q = f^{(n)}(x_1).$$

Writing $a + \theta(b-a)$, where $0 < \theta < 1$, for x_1, substituting the resulting value of Q in (1), and transposing, we get Taylor's Theorem as enunciated.

Taylor's Theorem is more usually quoted in the following form, obtained by writing x for b:—

$$\mathbf{f(x) = f(a) + (x-a)\,f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots}$$
$$\mathbf{+ \frac{(x-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(x-a)^n}{n!}f^{(n)}\{a + \theta(x-a)\}.} \quad (2)$$

But if we put h for $b - a$, and replace a by x, we get the form

$$\mathbf{f(x+h) = f(x) + h\,f'(x) + \frac{h^2}{2!}f''(x) + \dots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(x)}$$
$$\mathbf{+ \frac{h^n}{n!}f^{(n)}(x + \theta h).}$$

Putting $a = 0$ in (2), we get Maclaurin's Theorem:—

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{x^n}{n!}f^{(n)}(\theta x).}$$

6·91. Taylor's Series. If we denote the first n terms on the right of equation (2) above by $S_n(x)$ and the

$(n+1)$th, i.e. the last, term by $R_n(x)$, Taylor's theorem can be written as

$$f(x) = S_n(x) + R_n(x).$$

Under certain conditions (see Chap. XV) we may take the limit of this as $n \to \infty$. If, moreover, $R_n(x) \to 0$ as $n \to \infty$, we get

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots$$

This is the same series as that of § 63.

If the function to be expanded is algebraic and rational, and of degree m (say), $f^{(m+1)}(x)$ and all the succeeding differential coefficients vanish. So the expansion is finite, and questions of convergency do not arise.

EXAMPLES ON CHAPTER VI

1. Apply Taylor's theorem to express $\sin(\frac{1}{4}\pi + \theta)$ in a series of powers of θ. [*Benares*, 1934]

2. Prove that $f(mx)$ is equal to

$$f(x) + (m-1)xf'(x) + (1/2!)(m-1)^2x^2f''(x) + (1/3!)(m-1)^3x^3f'''(x) + \dots$$

[Write $f(mx)$ as $f\{x + (m-1)x\}$ and apply Taylor's Theorem.]

3. Show that $\log\cosh x = \frac{1}{2}x^2 - \frac{1}{12}x^4 + \frac{1}{45}x^6 - \dots$

4. Apply Maclaurin's theorem to find the expansion of $e^x/(e^x+1)$, as far as the term in x^3. [*Allahabad*, 1931]

5. Prove that if $\log_e y = \tan^{-1} x$,

$$(1+x^2)\frac{d^ny}{dx^n} = \{1 - 2(n-1)x\}\frac{d^{n-1}y}{dx^{n-1}} - (n-1)(n-2)\frac{d^{n-2}y}{dx^{n-2}},$$

and hence find the coefficient of x^r in the expansion of y by Maclaurin's theorem. [*Madras*, 1937]

6. Prove that for all finite values of x,

$$e^x \sin x = x + x^2 + \frac{2}{3!}x^3 - \frac{2^2}{5!}x^5 - \dots + \sin(\tfrac{1}{4}n\pi)\frac{2^{n/2}}{n!}x^n + \dots$$

[*Benares*, 1933]

7. Prove that

$$\frac{d^n}{dx^n}e^{x\cos\alpha}\cos(x\sin\alpha) = e^{x\cos\alpha}\cos(x\sin\alpha + n\alpha).$$

Hence show that

$$e^{x \cos a} \cos (x \sin a) = 1 + x \cos a - \frac{x^2}{2!} \cos 2a + \frac{x^3}{3!} \cos 3a + \ldots$$

[*Bombay*, 1936]

8. If $y = \sin \log (x^2 + 2x + 1)$,
prove that

$$(x + 1)^2 y_{n+2} + (2n + 1)(x + 1) y_{n+1} + (n^2 + 4) y_n = 0.$$

Hence or otherwise expand y in ascending powers of x as far as x^6. [*Agra*, 1935]

9. Show, if $y = \sin (m \sin^{-1} x)$, that

$$(1 - x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} + m^2 y = 0.$$

Hence or otherwise expand $\sin m\theta$ in powers of $\sin \theta$.

Show that $e^{\theta} = 1 + \sin \theta + \frac{1}{2!} \sin^2 \theta + \frac{2}{3!} \sin^3 \theta + \ldots$

[*Lucknow*, 1933]

10. Expand $e^{a \sin^{-1} x}$ by Maclaurin's theorem and find the general term. [*Allahabad*, 1932]

11. Expand $\log \{1 + \log (1 + x)\}$ in powers of x by Maclaurin's theorem as far as the term in x^3.

By substituting $x/(1 + x)$ for x deduce the expansion of

$$\log \{1 + \log (1 + x)\}$$

as far as the term in x^3.

12. If $y = (\sin^{-1} x) \sqrt{(1 - x^2)}$,
where $-1 < x < 1$, and $-\frac{1}{2}\pi < \sin^{-1} x < \frac{1}{2}\pi$, prove that

$$(1 - x^2) \frac{d^{n+1}y}{dx^{n+1}} - (2n - 1) x \frac{d^n y}{dx^n} - n^2 \frac{d^{n-1}y}{dx^{n-1}} = 0.$$

Assuming that y can be expanded in ascending powers of x in the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

prove that $(n + 1) a_{n+1} = n a_{n-1}$,
and hence obtain the general term of the expansion.

[*Math. Tripos*, 1933]

13. By Maclaurin's Theorem or otherwise find the expansion of

$$y = \sin (e^x - 1)$$

up to and including the terms in x^4.

Find also the first two non-vanishing terms in the expansion of x as a series of ascending powers of y. [*Math. Tripos*, 1926]

14. It is given that y is the positive value of

$$(1 + x + x^2)^{1/x^2}.$$

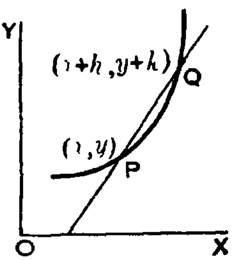
By means of the expansions of $\log_e (1 + x)$ and e^x, prove that when x is small

$$y = e^{1/2 + 1/x} (1 - \tfrac{3}{8} x + \ldots).$$ [*Math. Tripos*, 1924]

CHAPTER VII

TANGENTS AND NORMALS

7·1. Definition of Tangent. Let P be a given point on a curve and Q any other point on it. As Q tends to P, the straight line PQ tends, in general, to a definite straight line (whether Q be taken on one side of P or the other). This straight line is called the *tangent* to the curve at P.



7·11. Remark. The term "tends to" has been defined so far only in connection with variables which take up numerical values. Here this term has been used for positions. But there is no real difficulty. To "Q tends to P" we can give the meaning "the abscissa of Q tends to the abscissa of P" and to "PQ tends to a definite straight line" also we can give a similar meaning. For, because PQ always passes through the given point P, the definite straight line to which it tends must also pass through P, and therefore to "PQ tends to PT" we can assign the meaning that "the angle between PQ and the x-axis tends to the angle between PT and the x-axis."

7·12. Equation of the Tangent. Let the given point P on the curve $y = f(x)$ be (x_1, y_1), and let Q, any other point on the curve, be $(x_1 + h, y_1 + k)$. Then the straight line through P and Q has the equation

$$y - y_1 = \frac{y_1 + k - y_1}{x_1 + h - x_1}(x - x_1),$$

or

$$y - y_1 = (k/h)(x - x_1).$$

Now, as Q tends to P, k/h tends to the value of dy/dx at (x_1, y_1); § 4·2.

Hence the equation of the tangent at (x_1, y_1) is

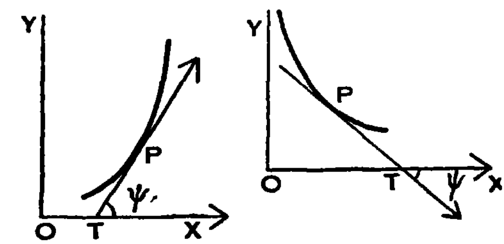
$$y - y_1 = f'(x_1)(x - x_1).$$

Usually the suffixes are dropped, the current coordinates are represented by X, Y and the result is quoted as follows:

the tangent to the curve $y = f(x)$ at the point (x, y) is

$$Y - y = \frac{dy}{dx}(X - x).$$

7·13. Geometrical meaning. Let the positive direction of the tangent be defined as the one in which those points of the curve lie whose abscissae are greater than that of P. Let ψ be the angle (positive, zero or negative, but not reflex), between the positive direction of the tangent at (x, y) and the positive direction of the x-axis (see the accompanying figures).



Comparing the equation of the tangent obtained above with the standard equation $y = mx + c$ of a straight line, we see that

$$\frac{dy}{dx} = \tan\psi,$$

i.e., *the differential coefficient dy/dx at the point (x, y) is equal to the trigonometrical tangent of the angle which the positive direction of the tangent to the curve at (x, y) makes with the positive direction of the x-axis.*

It is clear from the figures that $\tan\psi$ (and therefore dy/dx) is positive when y increases with x, and negative when y diminishes as x increases. (*Cf.* § 4·31).

A little consideration will show that ψ must lie between $+\frac{1}{2}\pi$ and $-\frac{1}{2}\pi$.

The tangent of the angle between the tangent at P to the curve and the x-axis is usually called the **gradient** of the curve at the point P.

Ex. 1. Find the tangent at (1, 2) to the curve $y = x^3 + 1$.

Here $dy/dx = 3x^2$, and the value of dy/dx at (1, 2) is 3. Hence the required tangent is $y - 2 = 3(x - 1)$.

Ex. 2. Find the equation of the tangent at (x, y) on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Differentiating, $$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0.$$

Therefore $$\frac{dy}{dx}=-\frac{b^2x}{a^2y}.$$

The equation to the tangent at (x, y) is, therefore,

$$Y-y=-\frac{b^2x}{a^2y}(X-x),$$

or $$\frac{y}{b^2}(Y-y)+\frac{x}{a^2}(X-x)=0,$$

or $$\frac{Xx}{a^2}+\frac{Yy}{b^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Hence the tangent at (x,y) to $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\frac{Xx}{a^2}+\frac{Yy}{b^2}=1.$$

Important. The equation to the tangent must in every case be simplified as much as possible. Such simplification can always be carried out in the case of algebraic curves, as in the above example.

Ex. 3. Find the tangent at the point t on the curve

$$x=a\cosh t,\ y=b\sinh t.$$

Here $$\frac{dy}{dx}=\left(\frac{dy}{dt}\right)\Big/\left(\frac{dx}{dt}\right)=\frac{b\cosh t}{a\sinh t}.$$

Hence the tangent at t is

$$Y-b\sinh t=\frac{b\cosh t}{a\sinh t}(X-a\cosh t);$$

i.e., $$\begin{aligned}(b\cosh t)X-(a\sinh t)Y &= ab(\cosh^2 t-\sinh^2 t)\\ &= ab.\end{aligned}$$

7·14. Angle of intersection of curves. The angle between the two straight lines

$$y=m_1x+c_1 \text{ and } y=m_2x+c_2$$

is known from coordinate geometry to be

$$\tan^{-1}\{(m_1 \sim m_2)/(1 + m_1 m_2)\}.$$

If we replace m_1 and m_2 in this formula by the values of dy/dx for the two curves at any of their points of intersection, we get at once the angle at which the curves cut there.

Ex. Find the angle of intersection of the curves

$$y = 4 - x^2, \quad \ldots \ldots \ldots \quad (1)$$

$$\text{and } y = x^2. \quad \ldots \ldots \ldots \quad (2)$$

Subtracting, the abscissa of the point of intersection is given by

$$4 - 2x^2 = 0.$$

Therefore $\qquad x = \sqrt{2}.$

(1) gives $dy/dx = -2x = -2\sqrt{2}$ at the point of intersection, and (2) gives $dy/dx = 2x = 2\sqrt{2}$ at the point of intersection.

Hence, if θ is the required angle of intersection,

$$\tan\theta = -\frac{2\sqrt{2} + 2\sqrt{2}}{1 - 4 \cdot 2} = \frac{4\sqrt{2}}{7}.$$

Therefore $\qquad \theta = \tan^{-1}(4\sqrt{2}/7).$

EXAMPLES

1. Find the equation of the tangent at the point (x, y) on each of the following curves :—

(i) $y^2 = 4ax$. $\qquad$ (ii) $xy = a^2$.

(iii) $x^m/a^m + y^m/b^m = 1$. $\qquad$ (iv) $(x^2 + y^2)^2 = a^2(x^2 - y^2)$.

(v) $y = a\cosh(x/a)$. $\qquad$ (vi) $y = a\log\sin x$.

2. Find the points at which the tangent to each of the following curves is (a) parallel to and (b) perpendicular to the axis of x:

(i) $ax^2 + 2hxy + by^2 = 1$,

(ii) $xy = 2a(x^2 + a^2)$,

✓(iii) $y = a + \log x + b^3/x$.

[The tangent is perpendicular to the x-axis if $dx/dy = 0$.]

3. Find the equation of the tangent at the point t on each of the following curves :

(i) $x = t^2 - a,\ y = t^3 - b$,

(ii) $x = a(t + \sin t),\ y = a(1 - \cos t)$,

✓(iii) $x = a\sin^3 t,\ y = b\cos^3 t$.

4. Prove that $x/a + y/b = 1$ touches the curve $y = b\,e^{-x/a}$ at the point where the curve crosses the axis of y. [*Madras*, 1936]

5. Prove that the curve $(x/a)^n + (y/b)^n = 2$ touches the straight line $x/a + y/b = 2$ at the point (a, b), whatever be the value of n.

6. Prove that all points of the curve

$$y^2 = 4a\,\{x + a\sin(x/a)\}$$

at which the tangent is parallel to the axis of x lie on a parabola. [*Patna*, 1935]

7. Tangents are drawn from the origin to the curve $y = \sin x$. Prove that their points of contact lie on

$$x^2y^2 = x^2 - y^2. \qquad \text{[Lucknow, 1931]}$$

8. Show that the tangents to the Folium of Descartes $x^3 + y^3 = 3axy$ at the points where it meets the parabola $y^2 = ax$ are parallel to the axis of y. [*Benares*, 1931]

9. If $p = x\cos\alpha + y\sin\alpha$ touch the curve

$$\left(\frac{x}{a}\right)^{n/(n-1)} + \left(\frac{y}{b}\right)^{n/(n-1)} = 1,$$

prove that

$$p^n = (a\cos\alpha)^n + (b\sin\alpha)^n. \qquad \text{[Patna, 1931]}$$

10. Find the angle of intersection of the curves

(i) $2y^2 = x^3$ and $y^2 = 32x$,

(ii) $x^2 = 4ay$ and $2y^2 = ax$,

(iii) $xy = a^2$ and $x^2 + y^2 = 2a^2$.

11. Show that the condition that the curves

$$ax^2 + by^2 = 1 \text{ and } a'x^2 + b'y^2 = 1$$

should intersect orthogonally is that

$$\frac{1}{a} - \frac{1}{b} = \frac{1}{a'} - \frac{1}{b'}.$$

12. In the curve $x^m y^n = a^{m+n}$, prove that the portion of the tangent, intercepted between the axes, is divided at its point of contact into segments which are in a constant ratio. [*Agra*, 1935]

7·2. Normal. The *normal* to a curve at any point is the straight line which passes through that point and is at right angles to the tangent to the curve at that point.

Any line through the point (x, y) is

$$Y - y = m(X - x).$$

This will be perpendicular to the tangent to the curve at (x, y) if

$$m \cdot \frac{dy}{dx} = -1.$$

Substituting the value of m obtained from this in the first equation, we see that

the normal at (x, y) to the curve $y = f(x)$ is

$$\mathbf{\frac{dy}{dx}(Y - y) + (X - x) = 0.}$$

Ex. Find the equation of the normal at (x, y) to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

As before (Ex. 2, p. 88), $\dfrac{dy}{dx} = -\dfrac{b^2x}{a^2y}$.

Hence the normal at (x, y) is

$$-\frac{b^2x}{a^2y}(Y - y) + (X - x) = 0,$$

or

$$\frac{X - x}{x/a^2} = \frac{Y - y}{y/b^2}.$$

Examples

1. Find the equation of the normal at (x, y) on each of the curves of Ex. 1, p. 89.

2. Find the equations of the tangent and normal to the curve

$$y(x-2)(x-3) - x + 7 = 0$$

at the point where it cuts the axis of x. [*Dacca*, 1936]

3. Find the equation of the normal at the point (x, y) on each of the curves of Ex. 3, p. 89.

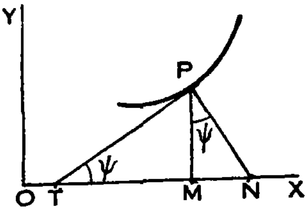
4. Find the tangent and normal to the curve $y^2 = 4ax$ and find the length of the normal chord at a point (x', y'). [*Lucknow*, 1933]

5. If the normal to the curve $x^{2/3} + y^{2/3} = a^{2/3}$ makes an angle ϕ with the axis of x, show that its equation is

$$y \cos\phi - x \sin\phi = a \cos 2\phi. \qquad \text{[Nagpur, 1931]}$$

6. In the catenary $y = a \cosh(x/a)$, prove that the length of the portion of the normal intercepted between the curve and the axis of x is y^2/a.

7·21. Cartesian Sub-tangent and Sub-normal. Let the tangent and the normal at any point P on a curve meet the x-axis in T and N respectively, and let PM be the ordinate. Then TM is called the *sub-tangent* and MN the *sub-normal*.



If the angle which the tangent makes with the x-axis be ψ, we have $\tan\psi = dy/dx$ and $MP = y$. Therefore

$$\textbf{Sub-tangent} = TM = y\cot\psi = y \Big/ \left(\frac{dy}{dx}\right);$$

$$\textbf{Sub-normal} = MN = y\tan\psi = y\frac{dy}{dx}.$$

7·22. Intercepts. The equation to the tangent is

$$Y - y = (dy/dx)(X - x).$$

This meets OX where

$$0 - y = \frac{dy}{dx}(X - x), \quad \text{or} \quad X = x - y \Big/ \left(\frac{dy}{dx}\right).$$

Hence the *intercept which the tangent cuts off from the axis of x is*

$$x - y \Big/ \left(\frac{dy}{dx}\right).$$

The intercept in the figure of the previous article is OT. Similarly, by putting $X = 0$ in the equation of the tangent, we see that the *intercept which the tangent cuts off from the axis of y is*

$$y - x\frac{dy}{dx}.$$

The length of the tangent intercepted between the point of contact and the axis of x is often called *the length of the tangent*. In the figure this is PT and its value is

$$y\operatorname{cosec}\psi = y\sqrt{(1 + \cot^2\psi)} = \frac{y\sqrt{(1 + \tan^2\psi)}}{\tan\psi}$$

$$= y\frac{\sqrt{\{1 + (dy/dx)^2\}}}{dy/dx}.$$

Similarly, by *the length of the normal* is understood PN, the value of which is $y\sec\psi$, i.e., $y\sqrt{\{1 + (dy/dx)^2\}}$.

7·23. Sign. In the first figure on page 87, y and dy/dx are both positive (see § 7·13). Hence $y/(dy/dx)$ will be positive in such a case. The subtangent, therefore, is measured positively from T in the direction OX. If its value comes out to be negative, it indicates that M lies to the left of T. A similar interpretation is possible in the case of the subnormal and the intercepts.

EXAMPLES

1. Show that in the exponential curve $y = be^{x/a}$, the subtangent is of constant length, and the subnormal varies as the square of the ordinate. [*Patna*, 1937]

2. Show that the subtangent at any point of the curve $x^m y^n = a^{m+n}$ varies as the abscissa. [*Benares*, 1928]

3. Show that in the case of the curve $\beta y^2 = (x + \alpha)^3$, the square of the subtangent varies as the subnormal. [*Allahabad*, 1937]

4. Find the lengths of the normal and subnormal to the curve $y = \frac{1}{2}a(e^{x/a} + e^{-x/a})$. [*Punjab*, 1932]

5. In the curve $x^{m+n} = a^{m-n} y^{2n}$, prove that the mth power of the subtangent varies as the nth power of the subnormal.

6. Find the subtangent, subnormal, normal, tangent, and the intercept on the axis at the point t on the cycloid.

$$x = a(t + \sin t), \quad y = a(1 - \cos t).$$

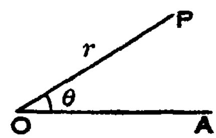
7. What should be the value of n in the equation to the curve $y = a^{1-n} x^n$ in order that the subnormal may be of constant length?

8. Prove that for the catenary $y = c \cosh (x/c)$, the perpendicular dropped from the foot of the ordinate upon the tangent is of constant length.

9. If x_1, y_1 be the parts of the axes of x and y intercepted by the tangent at any point (x, y) on the curve $(x/a)^{2/3} + (y/b)^{2/3} = 1$, show that $x_1^2/a^2 + y_1^2/b^2 = 1$. [*Agra*, 1934]

10. Prove that in the ellipse $x^2/a^2 + y^2/b^2 = 1$, the length of the normal varies inversely as the perpendicular from the origin on the tangent.

7·3. Polar Coordinates. The position of a point P on a plane can also be indicated by stating (i) its distance r from a fixed point O, and (ii) the inclination θ of OP to a fixed straight line OA. r and θ are called the *polar coordinates* of P. r is called the *radius vector*, θ the *vectorial angle*, O the *pole*, and OA the *initial line*. r is considered to



be positive when measured away from O along the line bounding the

vectorial angle, and θ is considered to be positive when measured in the anti-clockwise direction.

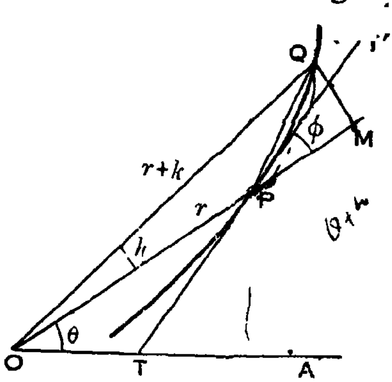
It is usual to regard θ as the independent variable.

When converting polar coordinates to Cartesians, or vice versa, it is customary to take the pole as the origin and the initial line as the x-axis. Then the formulæ of conversion are

$$x = r\cos\theta, \quad y = r\sin\theta.$$

If r and θ are given, there is only one point which will have the coordinates (r, θ). But if P be given, not only may the coordinates be stated as (r, θ), but also as $(r, \theta + 2\pi)$, $(r, \theta + 4\pi)$, ..., or as $(-r, \theta \pm \pi)$, $(-r, \theta + 3\pi)$, ... The student should remember this, otherwise he would commit mistakes.

7·4. Angle between Radius Vector and Tangent.



Let P be a given point on the curve $r = f(\theta)$ and let Q be any other point. Let P be the point (r, θ) and Q the point $(r + k, \theta + h)$*.

Let TPT' be the tangent at P, and let φ be the angle between it and the radius vector OP. We have to find φ.

Draw QM perpendicular to OP (produced if necessary). Then as $h \to 0$, $Q \to P$, the secant $PQ \to$ the tangent PT' and the $\angle QPM \to \angle\varphi$.

Hence $\tan\varphi = \tan\lim_{h\to 0}\angle QPM = \lim_{h\to 0}\tan QPM$

$$= \lim_{h\to 0}\frac{QM}{PM} = \lim_{h\to 0}\frac{(r+k)\sin h}{(r+k)\cos h - r}$$

*In the equation $r = f(\theta)$, r and θ are the current coordinates; but when we say that the coordinates of the given point are r and θ, we attach a different meaning to these symbols. It would be best to use r_1, θ_1 for the coordinates of P, like what was done in the case of the Cartesian equation (§ 7·12). But this is inconvenient. However, after this explanation the student should have no difficulty in understanding whether at any given place the reference is to the coordinates of P or to the current coordinates.

$$= \lim_{h \to 0} \frac{(r + k) \sin h}{k \cos h - r(1 - \cos h)}$$

$$= \lim_{h \to 0} \frac{(r + k) \dfrac{\sin h}{h}}{\dfrac{k}{h} \cos h - r \dfrac{2 \sin^2 \frac{1}{2}h}{h}}$$

Now $$\lim_{h \to 0} \frac{k}{h} = \frac{dr}{d\theta}. \qquad (\S\ 4 \cdot 2)$$

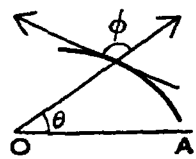
Also $\lim_{h \to 0} \dfrac{2 \sin^2 \frac{1}{2}h}{h} = \lim_{h \to 0} \dfrac{\sin \frac{1}{2}h}{\frac{1}{2}h} \sin \frac{1}{2}h = 1 \times 0 = 0,$

and $$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

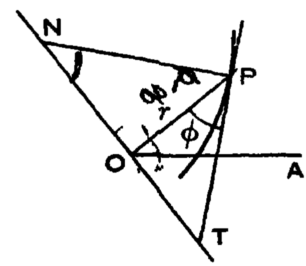
It follows that $$\tan \phi = \frac{(r + 0) \cdot 1}{\dfrac{dr}{d\theta} - r \cdot 0}$$

i.e., $$\tan \phi = r \frac{d\theta}{dr}.$$

If we suppose ϕ to be numerically less than π and define it to be the angle between the positive direction of the radius vector and that direction of the tangent in which θ increases, it is easy to see that the above formula holds whether $r d\theta/dr$ be positive or negative. If $r d\theta/dr$ be negative, it means that ϕ is greater than $\frac{1}{2}\pi$. (See the marginal figure).



7·41. Polar Subtangent and Subnormal. Let P be any point on a curve, and let the tangent and normal at P meet the straight line through the pole at right angles to the radius vector OP in T and N respectively.



Then OT is called the *polar subtangent* and ON the *polar subnormal*.

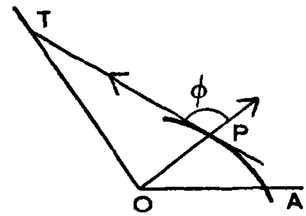
Since the angle OPT is ϕ, $OT = r \tan \phi$. Hence

$$\text{the } \textbf{polar subtangent} = \mathbf{r^2 \frac{d\theta}{dr}}.$$

Also, $\quad ON = r \tan \angle OPN = r \cot \varphi$. Hence

$$\text{the \textbf{polar subnormal}} = \frac{\mathbf{dr}}{\mathbf{d\theta}}.$$

The above formula shows that the polar subtangent is measured positively to the right, the observer being supposed to be stationed at O and looking in the direction of P. A negative value of the subtangent shows that T is to the left, as in the marginal figure. A similar meaning can be assigned to a negative subnormal.



Ex. Find the angle between the radius vector and the tangent at any point on the cardioid $r = a(1 - \cos \theta)$.

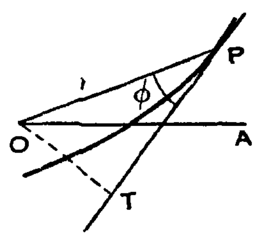
$$\text{Differentiating,} \qquad \frac{dr}{d\theta} = a \sin \theta.$$

$$\text{Therefore } \tan \phi = r\frac{d\theta}{dr} = \frac{a(1 - \cos \theta)}{a \sin \theta}$$

$$= \frac{2 \sin^2 \tfrac{1}{2}\theta}{2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta} = \tan \tfrac{1}{2}\theta.$$

$$\text{Therefore} \qquad \phi = \tfrac{1}{2}\theta.$$

7·42. Perpendicular from Pole on Tangent. If p be the length of the perpendicular OT from the pole O to the tangent at any point P on the curve, then, from the figure,



$$\mathbf{p = r \sin \varphi}.$$

If we want the result in terms of r and θ, we have only to substitute for $\sin \varphi$ from the equation $\tan \varphi = r\, d\theta/dr$.

$$\text{Thus} \qquad \frac{1}{p^2} = \frac{1}{r^2} \operatorname{cosec}^2 \varphi = \frac{1}{r^2}(\cot^2 \varphi + 1)$$

$$= \frac{1}{r^2}\left\{\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 + 1\right\},$$

$$\text{or} \qquad \frac{\mathbf{1}}{\mathbf{p^2}} = \frac{\mathbf{1}}{\mathbf{r^2}} + \frac{\mathbf{1}}{\mathbf{r^4}}\left(\frac{\mathbf{dr}}{\mathbf{d\theta}}\right)^2.$$

Sometimes u is used to denote $1/r$, and this formula is stated as

$$1/p^2 = u^2 + (du/d\theta)^2.$$

Ex. In the cardioid, $r = a(1 - \cos\theta)$, we have $p = r\sin\phi$ $= a(1 - \cos\theta).\sin\frac{1}{2}\theta$ (see Ex. above) $= 2a\sin^3\frac{1}{2}\theta$.

7·43. Pedal equation. The relation between p and r for a given curve is called its pedal equation. For certain curves this equation is very simple.

(i) To find the pedal equation from the Cartesian equation.

The tangent at (x, y) is

$$Y - y - (X - x)\frac{dy}{dx} = 0.$$

Hence, if the perpendicular on it from the origin ($X = 0$, $Y = 0$) is p, we have

$$p = \frac{x\frac{dy}{dx} - y}{\sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}}. \quad . \quad . \quad . \quad . \quad (1)$$

Also $\qquad r^2 = x^2 + y^2, \quad . \quad . \quad . \quad . \quad (2)$

and the equation to the curve is known, say

$$f(x, y) = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Hence we get the pedal equation by eliminating and y between the equations (1), (2) and (3).

(ii) To find the pedal equation from the polar equation. Let the equation to the curve be

$$f(r, \theta) = 0. \quad . \quad (1)$$

Also, by the previous articles,

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2. \quad . \quad (2)$$

We get the pedal equation by eliminating θ between (1) and (2).

Ex. 1. Find the pedal equation of the parabola $y^2 = 4a(x + a)$.

Differentiating, $\qquad 2y\frac{dy}{dx} = 4a$, or $\frac{dy}{dx} = \frac{2a}{y}$.

Hence the tangent at (x, y) is $Y - y = (2a/y)(X - x)$.

Therefore $$p = \frac{x \cdot 2a/y - y}{\sqrt{(1 + 4a^2/y^2)}} = \frac{2ax - y^2}{\sqrt{(y^2 + 4a^2)}}$$
$$= \frac{2ax - 4a(x + a)}{\sqrt{(4ax + 4a^2 + 4a^2)}} = \frac{-2a(x + 2a)}{\sqrt{\{4a(x + 2a)\}}}$$
$$= -\sqrt{\{a(x + 2a)\}}.$$

Also $$r^2 = x^2 + y^2 = x^2 + 4a(x + a) = (x + 2a)^2.$$

Therefore $$p^2 = ar,$$
which is the required pedal equation.

Ex. 2. Find the pedal equation of the cardioid $r = a(1 - \cos\theta)$. As before (Ex., § 7·42) we can show that
$$p = 2a \sin^3 \tfrac{1}{2}\theta.$$

Also $$r = a(1 - \cos\theta) = 2a \sin^2 \tfrac{1}{2}\theta.$$

Therefore $$r^3 = 2ap^2,$$
which is the required pedal equation.

7·44. Angle of intersection of curves. If two curves, whose polar equations are known, intersect at P, and the values of ϕ at that point for the two curves be ϕ_1 and ϕ_2 respectively, then the angle of intersection of the two curves is evidently $\phi_1 \sim \phi_2$. If $\tan\phi_1 = n_1$ and $\tan\phi_2 = n_2$, then the angle of intersection is
$$\tan^{-1}\{(n_1 \sim n_2)/(1 + n_1 n_2)\}.$$
In particular, the curves cut orthogonally (i.e., at right angles) if
$$n_1 n_2 = -1.$$

EXAMPLES

1. Find the angle at which the radius vector cuts the curve $l/r = 1 + e\cos\theta$. [*Andhra*, 1937]

2. Find the angle of intersection of the curves $r = \sin\theta + \cos\theta$ and $r = 2\sin\theta$.

3. Find the angle of intersection of the curves $r = 2\sin\theta$ and $r = 2\cos\theta$.

4. Show that in the equiangular spiral $r = ae^{\theta\cot\alpha}$ the tangent is inclined at a constant angle to the radius vector.

5. Find the angle between the tangent and the radius vector in the case of the curve $r^n = a^n \sec(n\theta + \alpha)$, and prove that this curve is intersected by the curve $r^n = b^n \sec(n\theta + \beta)$ at an angle which is independent of a and b.

6. Find the angle ϕ for the curve
$$a\theta = (r^2 - a^2)^{1/2} - a\cos^{-1}(a/r).$$

7. Show that in the curve $r = a\theta$ the polar subnormal is constant and in the curve $r\theta = a$ the polar subtangent is constant.

8. Find the polar subtangent for the ellipse $l/r = 1 + e\cos\theta$.

9. For the cardioid $r = a(1 - \cos\theta)$ prove that (i) $\phi = \frac{1}{2}\theta$, (ii) $2ap^2 = r^3$, and (iii) the polar subtangent $= 2a\sin^2\frac{1}{2}\theta\tan\frac{1}{2}\theta$.

[*Benares*, 1934]

10. Find the value of the perpendicular from the pole upon the tangent to the curve $r(\theta - 1) = a\theta^2$.

11. Prove that the length of the perpendicular from the pole on the tangent to the ellipse $l/r = 1 + e\cos\theta$ is given by

$$\frac{1}{p^2} = \frac{1}{l^2}\left(\frac{2l}{r} - 1 + e^2\right).$$

12. Prove that, in the parabola $2a/r = 1 - \cos\theta$,

(i) $\phi = \pi - \frac{1}{2}\theta$,

(ii) $p = a\operatorname{cosec}\frac{1}{2}\theta$,

and (iii) the polar subtangent $= 2a\operatorname{cosec}\theta$.

13. Show that the pedal equation

(i) of the hyperbola $r^2\cos 2\theta = a^2$ is $pr = a^2$,

(ii) of the lemniscate $r^2 = a^2\cos 2\theta$ is $r^3 = a^2p$,

(iii) of the Archimedean spiral $r = a\theta$ is $p^2 = r^4/(r^2 + a^2)$;

(iv) of the sine spiral $r^n = a^n\sin n\theta$ is $pa^n = r^{n+1}$.

14. Show that the pedal equation of the ellipse

$$x^2/a^2 + y^2/b^2 = 1$$

is

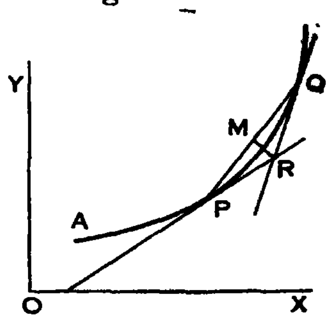
$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}.$$

15. Prove that the locus of the extremity of the polar subnormal of the curve $r = f(\theta)$ is $r = f'(\theta - \frac{1}{2}\pi)$.

Hence show that the locus of the extremity of the polar subnormal of the equiangular spiral $r = ae^{n\theta}$ is another equiangular spiral.

[*Lucknow*, 1934]

7·5. Differential coefficient of length of arc. Let the length of the arc AP of a curve, measured from a fixed point A on it be s. Then s evidently is some function, say $g(x)$, of x.



The problem is to find ds/dx when we know only the equation to the curve.

Let P be the point (x, y) and let Q, any other point on the

curve, be $(x+h, y+k)$. Let the tangents at P and Q meet in R and let RM be the perpendicular from R on PQ. Let the length of the arc AQ be equal to $s+\sigma$, so that the arc $PQ = \sigma$.

We shall assume as an axiom that

$$\text{chord } PQ < \sigma < PR + RQ. \quad . \quad . \quad . \quad (1)$$

Now $PR + RQ = PM \sec P + MQ \sec Q$, where P denotes the angle MPR and Q the angle MQR.

Thus $PR + RQ$

$$= PM\ (\sec P - 1) + MQ\ (\sec Q - 1) + PM + MQ.$$

Substituting this value of $PR + RQ$ in (1) and dividing by h we get

$$\frac{PQ}{h} < \frac{\sigma}{h} < \frac{PM}{h}(\sec P - 1) + \frac{MQ}{h}(\sec Q - 1) + \frac{PQ}{h}. \quad ..(2)$$

But $PQ^2 = (x+h-x)^2 + (y+k-y)^2 = h^2 + k^2$.

$$\text{Therefore } \lim_{h\to 0} \frac{PQ}{h} = \lim_{h\to 0} \frac{\sqrt{(h^2+k^2)}}{h}$$

$$= \lim_{h\to 0} \sqrt{\left(1 + \frac{k^2}{h^2}\right)}$$

$$= \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} \text{ by § 4·2.}$$

Also as $h \to 0$, i.e., as the point Q tends to the point P, PQ tends to the tangent PR, i.e., $\angle P \to 0$. Therefore $\sec P - 1 \to 0$. Moreover PM/h does not tend to infinity, because PQ/h does not tend to infinity (see above).

$$\text{Thus} \qquad \lim_{h\to 0} \frac{PM}{h}(\sec P - 1) = 0.$$

Again, as the point Q tends to the point P, the tangent at Q tends to the tangent at P. Therefore the exterior angle $R \to 0$. But, because $\angle P \to 0$, this means that $\angle Q$ also $\to 0$. Thus, we have also

$$\lim_{h\to 0} \frac{MQ}{h}(\sec Q - 1) = 0.$$

Also, as is well known, when we take limits, an inequality might have to be replaced by an equality. (Consider what happens when we take the limits, as $h \to 0$, of the quantities in $1 + h^2 < 1 + 2h^2$.)

Finally, $$\lim_{h\to 0} \frac{\sigma}{h} = \frac{ds}{dx} \text{ by § 4·2.}$$

Thus from (2) we get, upon taking the limits of the quantities involved,

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \leqslant \frac{ds}{dx} < \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

This shows that $\dfrac{ds}{dx} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$.

7·51. Corollary. We have proved above that

$$\lim_{h\to 0} PQ/h = \lim_{h\to 0} \sigma/h,$$

from which we can infer at once that $\lim_{h\to 0} PQ/\sigma = 1$,

i.e., $$\lim_{h\to 0} (\text{chord } PQ/\text{arc } PQ) = 1.$$

7·52. Alternative Proof of the Formula for ds/dx. If we *assume* that

$$\lim_{h\to 0} \frac{\text{chord } PQ}{\text{arc } PQ} = 1, \quad . \quad . \quad . \quad . \quad (1)$$

we can deduce the value of ds/dx more easily as follows:

By (1) $$\lim_{h\to 0} \frac{\text{chord } PQ}{h} \cdot \frac{h}{\text{arc } PQ} = 1,$$

i.e., $$\lim_{h\to 0} \frac{\text{chord } PQ}{h} = \lim_{h\to 0} \frac{\text{arc } PQ}{h}.$$

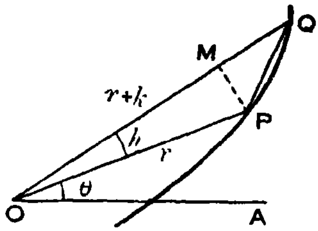
But the left hand side $= \lim_{h\to 0} \dfrac{\sqrt{(h^2 + k^2)}}{h}$

$$= \lim_{h\to 0} \sqrt{\left\{1 + \frac{k^2}{h^2}\right\}} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

and the right hand side $= \lim_{h\to 0} \dfrac{\sigma}{h} = \dfrac{ds}{dx}$.

Hence $$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

7·53. Differential coefficient of the arc. Polar formula. Let P be any point (r, θ) on the curve $r = f(\theta)$, and let Q, any other point on the curve, be $(r+k, \theta+h)$. Then s, the length of the arc up to P, measured from some fixed point on the curve, will be a function of θ, and



$$\frac{ds}{d\theta} = \lim_{h\to 0} \frac{\text{arc } PQ}{h}, \text{ by § 4·2}$$

$$= \lim_{h\to 0} \frac{\text{chord } PQ}{h}, \text{ by § 7·51.}$$

Now chord PQ^2

$= PM^2 + MQ^2$, where PM is perpendicular to OQ

$= r^2 \sin^2 h + (r + k - r\cos h)^2$

$= r^2 \sin^2 h + k^2 + 2rk(1 - \cos h) + r^2(1 - \cos h)^2.$

Therefore $\lim_{h\to 0} \dfrac{\text{chord } PQ}{h}$

$$= \sqrt{\left\{r^2 \frac{\sin^2 h}{h^2} + \frac{k^2}{h^2} + 2r\frac{k}{h} \cdot \frac{2\sin^2 \frac{1}{2}h}{h} + \frac{4r^2 \sin^4 \frac{1}{2}h}{h^2}\right\}}$$

$$= \sqrt{\left\{r^2 . 1 + \left(\frac{dr}{d\theta}\right)^2 + 2r . \left(\frac{dr}{d\theta}\right) . 0 . + 0\right\}}.$$

Thus
$$\frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}.$$

7·54. Other formulæ.

(1) We know that $dy/dx = \tan\psi$, where ψ is the angle which the tangent makes with the x-axis. Hence

$$\frac{ds}{dx} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} = \left\{1 + \tan^2\psi\right\}^{\frac{1}{2}} = \pm \sec\psi.$$

Therefore $$\frac{dx}{ds} = \cos \psi,$$

if s be measured in such a way that dx/ds is positive, i.e., if x and s increase together.

(2) Again,

$$\frac{dy}{ds} = \frac{dy}{dx} \cdot \frac{dx}{ds} = \tan \psi \cos \psi = \sin \psi,$$

if x, y and s increase together.

(3) If x and y are given in terms of a parameter, say $x = f_1(t)$, $y = f_2(t)$, then

$$\frac{ds}{dt} = \frac{ds}{dx} \cdot \frac{dx}{dt} = \left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{1}{2}} \cdot \frac{dx}{dt}$$

$$= \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2\left(\frac{dx}{dt}\right)^2\right\}^{\frac{1}{2}}.$$

Hence $$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

(4) We have seen that $\tan \phi = r d\theta/dr$.

Hence $$\cos \phi = \frac{1}{\left\{1 + r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{\frac{1}{2}}}$$

$$= \frac{\dfrac{dr}{d\theta}}{\left\{\left(\dfrac{dr}{d\theta}\right)^2 + r^2\right\}^{\frac{1}{2}}} = \pm \frac{\dfrac{dr}{d\theta}}{\dfrac{ds}{d\theta}};$$

i.e., $$\cos \phi = \frac{dr}{ds},$$

s being measured in such a way that s and θ increase together, so that ϕ is acute when r increases with s (see § 7·4).

Also $$\sin \phi = \tan \phi \cos \phi = \frac{r\,d\theta}{dr} \cdot \frac{dr}{ds} = \frac{r\,d\theta}{ds}.$$

Ex. For the cycloid $x = a(1 - \cos t)$, $y = a(t + \sin t)$, find ds/dt, ds/dx and ds/dy.

Here $$\frac{dx}{dt} = a \sin t, \quad \frac{dy}{dt} = a(1 + \cos t).$$

Therefore $$\frac{ds}{dt} = \left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}^{\frac{1}{2}}$$

$$= a\,\{\sin^2 t + 1 + 2\cos t + \cos^2 t\}^{1/2}$$
$$= a\,\{2(1 + \cos t)\}^{1/2}$$
$$= a\,(4\cos^2 \tfrac{1}{2}t)^{1/2} = 2a \cos \tfrac{1}{2}t.$$

$$\frac{ds}{dx} = \frac{ds}{dt} \cdot \frac{dt}{dx} = 2a \cos \tfrac{1}{2}t / a \sin t = \operatorname{cosec} \tfrac{1}{2}t,$$

and $$\frac{ds}{dy} = \frac{ds}{dt} \cdot \frac{dt}{dy} = 2a \cos \tfrac{1}{2}t / 2a \cos^2 \tfrac{1}{2}t = \sec \tfrac{1}{2}t.$$

Examples

1. Calculate ds/dx for the following curves:

   (i) $y = ax^2 + bx + c$,

   (ii) $y = \log \cos x$,

   (iii) $y = a \cosh (x/a)$.

2. For the parabola $y^2 = 4ax$, prove that
$$\frac{ds}{dx} = \left(1 + \frac{a}{x}\right)^{\frac{1}{2}}.$$

3. For the ellipse $x = a \cos t$, $y = b \sin t$, prove that
$$\frac{ds}{dt} = a\,(1 - e^2 \cos^2 t)^{1/2}.$$

4. Calculate ds/dt for the following curves:

   (i) $x = t^2, y = t - 1$,

   (ii) $x = a \sec t$, $y = b \tan t$,

   (iii) $x = 2 \sin t$, $y = \cos 2t$.

5. Calculate $ds/d\theta$ for the following curves:

   (i) $r = \log \sin 3\theta$, (ii) $r = \frac{1}{2} \sec^2 \theta$.

6. For the curve $r = ae^{\theta \cot \alpha}$, prove that $s/r =$ constant, s being measured from the origin. [*Mysore*, 1936]

Examples on Chapter VII

1. Show that the abscissæ of the points on the curve $y = x(x - 2)(x - 4)$ where the tangents are parallel to the axis of x are given by $x = 2 \pm 2/\sqrt{3}$.

2. Find the points on the curve $y = x/(1 - x^2)$ where the tangent is inclined at angle of $\frac{1}{4}\pi$ to the x-axis.

3. Find the angle of intersection of the curves $x^2 - y^2 = a^2$ and $x^2 + y^2 = a^2\sqrt{2}$.

4. Find that normal to $\sqrt{(xy)} = a + x$ which makes equal intercepts upon the coordinate axes. [*Calcutta*, 1938]

5. Prove that in the curve $x^{2/3} + y^{2/3} = a^{2/3}$, the intercepts, made by the tangent at any point, on the coordinate axes are $a^{2/3}x^{1/3}$, $a^{2/3}y^{1/3}$ respectively.

Hence verify that the length of the tangent intercepted by the axes is constant. [*Bombay*, 1936]

6. In the tractrix

$$x = a(\cos t + \log \tan \tfrac{1}{2}t),$$
$$y = a \sin t,$$

prove that the portion of the tangent intercepted between the curve and the axis of x is of constant length. [*Allahabad*, 1931]

7. Find the abscissa of the point on the curve $ay^2 = x^3$ the normal at which cuts off equal intercepts from the coordinate axes.

8. Show that in the curve $y = a \log (x^2 - a^2)$ the sum of the tangent and the subtangent varies as the product of the coordinates of the point. [*Agra*, 1937]

9. Prove that the curves $y = e^{-ax} \sin bx$, $y = e^{-ax}$ touch at the points for which $bx = 2m\pi + \frac{1}{2}\pi$, where m is an integer. [*Lamb*]

10. If ϕ be the angle between the tangent to a curve and the radius vector drawn from the origin of coordinates to the point of contact, prove that

$$\tan \phi = \left(x\frac{dy}{dx} - y\right) \Big/ \left(x + y\frac{dy}{dx}\right).$$

11. Prove that the condition that $x \cos \alpha + y \sin \alpha = p$ should touch $x^m y^n = a^{m+n}$

is $$p^{m+n} \,.\, m^m \,.\, n^n = (m + n)^{m+n} a^{m+n} \cos^m \alpha \sin^n \alpha.$$

12. Show that the pedal equation of the spiral $r = a \operatorname{sech} n\theta$ is of the form

$$\frac{1}{p^2} = \frac{A}{r^2} + B,$$

and that the pedal equations of the curves $r = a e^{m\theta}$, $r\theta = a$, $r \sin n\theta = a$ and $r \cosh \theta = a$ are all of the same form.

13. Find the angle of intersection of the parabolas

$$r = a/(1 + \cos \theta) \text{ and } r = b/(1 - \cos \theta).$$

14. Find the angle of intersection of the cardioids

$$r = a(1 + \cos \theta) \text{ and } r = b(1 - \cos \theta).$$

15. Prove that the spirals $r^n = a^n \cos n\theta$ and $r^n = b^n \sin n\theta$ intersect orthogonally.

16. In the curve $r^m = a^m \cos m\theta$, prove that

$$ds/d\theta = a \sec^{(m-1)/m} m\theta.$$

[*Agra*, 1934]

17. Show that the normal to $y^2 = 4ax$ touches the curve

$$27ay^2 = 4(x - 2a)^3.$$

[Any normal to the parabola is $Y = mX - 2am - am^3$. (1)

The tangent at (x, y) to the curve $27ay^2 = 4(x - 2a)^3$ . . (2)

is $$Y - y = \frac{dy}{dx}(X - x),$$

or $$Y = \frac{dy}{dx} X + y - x\frac{dy}{dx}. \quad \text{. . . . . (3)}$$

To prove the proposition, we have to prove that (3) is the same as (1). Now choose the point of contact (x, y) in (3) in such a way that the coefficients of X in (1) and (3) become the same, i.e., let

$$\frac{dy}{dx} = m\text{; then } m = \frac{2(x - 2a)^2}{9ay} = \frac{3y}{2(x - 2a)}, \quad \text{. . (4)}$$

after division by $4(x - 2a)^3/27ay^2$ which is equal to 1 on account of (2).

Thus $$m^3 = \frac{27y^3}{8(x - 2a)^3} = \frac{y}{2a} \text{ by (2);}$$

i.e., $$y = 2am^3. \quad \text{. . . . . (5)}$$

Substituting this in (4) we get the value of x in terms of m.

Thus the coordinates x and y have been found in terms of m. Substituting these in the constant term of (3), viz., in $y - x.m$ we find that the latter reduces to $-2am - am^3$. Hence (3) is identical with (1). Therefore, etc.

This question can be solved more easily by the method of Envelopes. See Chapter XII.]

18. Show that the normal to the ellipse $x^2/a^2 + y^2/b^2 = 1$ touches the curve $(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$.

19. Prove that the equation to the tangent at the point determined by t on the curve $x = a\,\phi(t)/f(t)$, $y = a\,\psi(t)/f(t)$, may be written in the form

$$\begin{vmatrix} x & y & a \\ \phi(t) & \psi(t) & f(t) \\ \phi'(t) & \psi'(t) & f'(t) \end{vmatrix} = 0.$$

Obtain the slope of the curve at a point on the cycloid

$$x = a\,(t + \sin t),\ y = a\,(1 - \cos t).$$ [*Benares*, 1934]

20. The tangent at any point of the cardioid $r = a(1 + \cos\theta)$ whose vectorial angle is 2α meets the curve again at a point whose vectorial angle is 2β. Show that

$$\cos(2\beta - \alpha) + 2\cos\alpha = 0 \qquad [\textit{Agra, 1930}]$$

21. If r_1, r_2, denote the distances of any point P on the lemniscate $(x^2 + y^2)^2 = a^2(x^2 - y^2)$ from the points $(\pm a/\sqrt{2}, 0)$ and p_1, p_2, the perpendiculars on the tangent at P from these points, prove that

$$\frac{p_1}{r_1^2} + \frac{p_2}{r_2^2} = \sqrt{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right).$$

[*Lucknow*, 1935]

22. An equation $f(p, r) = 0$ connecting p and r is deduced from the equation $F(r, \theta) = 0$. Prove that if the equation $F(r, \theta) = 0$ is altered by writing r^n for r and $n\theta$ for θ, the equation $f(p, r) = 0$ must be altered by writing r^n for r and pr^{n-1} for p [*Math. Tripos*, 1927]

23. The rectangular coordinates of a point on a plane curve are given by

$$x = 3\cos\theta - \cos^3\theta, \qquad y = 3\sin\theta - \sin^3\theta.$$

Find the equation of the normal at any point on the curve and show that, at the point P where $\theta = \pi/4$, the normal passes through the origin. [*London*, 1933]

24. If P and Q are two points whose polar coordinates are (r, θ), $(r + \delta r, \theta + \delta\theta)$ respectively, and O is the pole, prove that

$$\cot OQP = \delta r/(r\sin\delta\theta) + \tan\tfrac{1}{2}\delta\theta \text{ accurately.}$$

Deduce the value of $\cot\phi$, where ϕ is the angle between the tangent and the radius vector at a point on the curve $r = f(\theta)$

Show that the tangent at the point, where $\theta = \pi/6$, on the curve $r = a\cos 2\theta$ meets the initial line at a distance $a/\sqrt{3}$ from the pole. [*London*, 1934]

MISCELLANEOUS EXAMPLES

1. Prove that $2^{1/x}$ is discontinuous at $x = 0$. Draw its graph.

2. Prove that $\lim_{x\to 0}\{1/(1+e^{1/x})\}$ does not exist.

3. Prove that $\lim_{x\to 0}\tan^{-1}(a/x^2)$ is $-\frac{1}{2}\pi$, 0, or $\frac{1}{2}\pi$ according as a is negative, zero, or positive.

4. Discuss the continuity of the functions f, ϕ and ψ, where

(i) $f(x) = x^2$, when $x < -2$,
$f(x) = 4$, when $-2 \leqslant x \leqslant 2$,
$f(x) = x^2$, when $x > 2$;

(ii) $\phi(x) = (x^3+1)/(x+1)$, when $x \neq -1$, $\phi(-1) = 0$;

(iii) $\psi(x) = (\sin^2 ax)/x^2$, when $x \neq 0$, $\psi(0) = 1$.

5. If n is a positive integer, prove that x^n is continuous for all values of x.

6. If $\phi(x) = x^2\sin(1/x)$, when $x \neq 0$, and $\phi(0) = 0$, show that $\phi'(x)$ is discontinuous at $x = 0$. [*Dacca*, 1936]

7. Differentiate $\tan^{-1} x^2$ from first principles.

8. Given that $\cos H = -\tan\phi\tan\delta$, where ϕ is a constant, show that

$$\sin 2\delta \cdot \frac{dH}{d\delta} + 2\cot H = 0. \qquad [\textit{Delhi}, 1935]$$

9. If $y = \dfrac{\sqrt{(x+1)} - 1}{\sqrt{(x+1)} + 1}$, show that

$$\frac{dy}{dx} = \frac{1}{\{1+\sqrt{(x+1)}\}^2\sqrt{(x+1)}}.$$

10. Prove that the differential coefficient of

$$\sqrt{\{x(a-x)\}} + \tfrac{1}{2}a\sin^{-1}\{(2x-a)/a\} \text{ is } \sqrt{(a/x - 1)}.$$

11. Show that each of the functions

$$\frac{1}{\sqrt{(a^2-b^2)}}\,\text{arc cos}\left(\frac{a\cos x + b}{a + b\cos x}\right),$$

$$\frac{2}{\sqrt{(a^2-b^2)}}\,\text{arc tan}\left\{\left(\frac{a-b}{a+b}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}x\right\},$$

has the derivative $1/(a + b\cos x)$.

12. In a table of logarithmic tangents to the base 10, show that the difference for one minute in the neighbourhood of 45° is approximately 0·0004.

13. Show that the equation

$$1/(x-a)+1/(x-b)+1/(x-c)=0$$

can have a pair of equal roots only if $a = b = c$.

[*Math. Tripos*, 1905]

14. By using the theorem

$$f(x+h)=f(x)+hf'(x),$$

find approximately the value of x^4+7x^3-2x+1 when $x=0·998$.

15. Apply Newton's method of approximating to the roots of an equation to the equation $x^2=3$, taking $\frac{7}{4}$ as the first approximation.

16. A man 6 feet high walks at a uniform rate of 4 miles per hour away from a lamp 20 feet high. Find the rate at which the length of his shadow increases.

17. A man on a wharf, 20 feet above the water-level, pulls in a rope to which a boat is attached at the rate of 4 feet per second. At what rate is the boat approaching the shore, when there is still 25 feet of rope out? [*Madras*, 1935]

18. A particle moves along a straight line and describes a distance s in time t. If $t=as^2+bs+c$, show that the particle undergoes a retardation which, at any instant, is proportional to the cube of the velocity. [*Andhra*, 1937]

19. Show how $f'(x)$ describes the rate of increase or decrease of $f(x)$. Illustrate this by $f(x)=\sin^{-1}x$ for $-\frac{1}{2}\pi<f(x)<\frac{1}{2}\pi$.

20. An angle is increasing at a constant rate. Show that the tangent increases eight times as fast as the sine when the angle is 60°.

21. The coordinates of three points P, Q, R on a curve are (1·2000, 2·3201), (1·2250, 2·3531), (1·2500, 2·3891) respectively. Calculate approximate values of dy/dx and d^2y/dx^2 at Q.

22. If $y=\sin(\sin x)$, prove that

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}\tan x+y\cos^2 x=0. \qquad [\textit{Dacca}, 1935]$$

23. If $y=x^2\cos x$, prove that

$$x^2\frac{d^2y}{dx^2}-4x\frac{dy}{dx}+(x^2+6)y=0 \qquad [\textit{Madras}, 1937]$$

24. If $y=x^4\log x$, prove that

$$\frac{d^6y}{dx^6}+\frac{24}{x^2}=0.$$

25. If $f(x) = e^{-x} \cos x$, show that

$$f^{iv}(x) + 4f(x) = 0.$$

26. If $y = e^{ax}\{a^2x^2 - 2nax + n(n+1)\}$, prove that

$$y_n = a^{n+2}\, x^2\, e^{ax}.$$

27. If $x^3 + y^3 - 3axy = 0$, show that

$$\frac{d^2y}{dx^2} = \frac{2a^3xy}{(ax - y^2)^3}.$$

28. Show that $y = \tan^{-1} x$ satisfies

$$(1 + x^2)y'' + 2xy' = 0.$$

Hence show how to find all the derivatives of y at $x = 0$.

[*Bombay*, 1936]

29. Prove that

$$\frac{d^n}{dx^n}\left(\frac{\log x}{x}\right) = (-1)^n\, n!\; x^{-n-1}\left\{\log x - 1 - \tfrac{1}{2} - \tfrac{1}{3} - \dots - \frac{1}{n}\right\}.$$

[*Punjab*, 1935]

30. Show that the nth differential coefficient of $1/(x^2 + x + 1)$ is

$$(-1)^n\, 2^{n+2}\, 3^{-(n+2)/2}\, (n\,!) \sin (n+1)\theta \sin^{n+1} \theta,$$

where $x = \{\cos (\theta + \frac{1}{3}\pi)\}/\sin \theta$.

31. Show that the nth differential coefficient of $\sin x \sinh x$ vanishes for the value $x = 0$ unless n be of the form $4m + 2$, when its value is $(-1)^m\, 2^{2m+1}$.

32. Prove that, if $y = (ax + b)/(cx + d)$, $2y'y''' = 3y''^2$; and that, if $a + d = 0$,

$$(y - x)y'' = 2y'(1 + y'). \qquad \text{[\textit{Math. Tripos}, 1911]}$$

33. The first, second, third and fourth differential coefficients of y with respect to x are denoted by t, a, b, c respectively, and those of x with respect to y by $\tau, \alpha, \beta, \gamma$. Show that

$$t^{-4}(3ac - 5b^2) = \tau^{-4}(3\alpha\gamma - 5\beta^2). \qquad \text{[\textit{Patna}, 1937]}$$

34. If u and v are two functions of x, possessing successive derivatives of desired orders, show that

$$v\frac{d^nu}{dx^n} = \frac{d^n}{dx^n}(uv) - n\frac{d^{n-1}}{dx^{n-1}}\left(u\frac{dv}{dx}\right) + \frac{n(n-1)}{1.2}\frac{d^{n-2}}{dx^{n-2}}\left(u\frac{d^2v}{dx^2}\right) - \dots$$

[*Dacca*, 1936]

35. Prove that

$$\frac{d^n}{dx^n}\left\{\frac{1}{x(x^2+1)}\right\}$$
$$= (-1)^n\, n\,!\,[x^{-n-1} - \cos\{(n+1)\cot^{-1} x\}(x^2+1)^{-(n+1)/2}].$$

36. Prove that

$$\log \frac{\tan x}{x} = \tfrac{1}{3}x^2 + \tfrac{7}{90}x^4 + \dots \qquad \text{[\textit{Allahabad}, 1938]}$$

37. Show that

$$\sinh x \cos x = x - \frac{2x^3}{3!} - \frac{2^2 x^5}{5!} + \dots$$

38. If $e^{e^x} = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$,
prove that

$$a_{n+1} = \frac{1}{n+1}\left\{a_n + \frac{a_{n-1}}{1!} + \frac{a_{n-2}}{2!} + \dots + \frac{a_{n-r}}{r!} + \dots + \frac{a_0}{n!}\right\},$$

and hence calculate the first six terms in the expansion.

39. Prove that at the origin the curve $y^2 = x^3$ touches the axis of x, $y^2 = x^3 + x$ touches the axis of y and $y^2 = x^2 + x^3$ bisects the angle between the axes.

40. Prove that the straight line $y = 3x - 1$ touches the curve

$$y = x^4 - 2x^2 + 3x$$

at two distinct points.

41. Show that the curves

$$x^3 - 3xy^2 = 2,$$
$$3x^2y - y^3 = 2,$$

cut orthogonally. [*Madras*, 1936]

42. Show that the curves $r = a(1 + \sin \theta)$ and $r = a(1 - \sin \theta)$ cut orthogonally.

43. For the curve $r^n = a^n \cos n\theta$, prove that

$$a^{2n}\frac{d^2r}{ds^2} + nr^{2n-1} = 0. \qquad \text{[\textit{Allahabad}, 1938]}$$

44. If the tangent to the curve $x^{1/2} + y^{1/2} = a^{1/2}$ at any point on it cuts the axes OX, OY at P, Q respectively, prove that $OP + OQ = a$. [*Madras*, 1935]

45. Show that the tangents at the points where the straight line $ax + hy = 0$ meets the ellipse

$$ax^2 + 2hxy + by^2 = 1$$

are parallel to the x-axis, and that the tangents at the points where the straight line $hx + by = 0$ meets the ellipse are parallel to the y-axis.

46. Prove that the normal at any point (r, θ) to the curve

$$r^n = a^n \cos n\theta$$

makes an angle $(n + 1)\theta$ with the initial line. [*Bombay*, 1936]

47. Show that the tangent at the point $(4am^2, 8am^3)$ of the curve $x^3 = ay^2$ meets the curve again in the point $(am^2, -am^3)$.

Show also that if $9m^2 = 2$, the tangent is also a normal to the curve. [*Andhra*, 1937]

48. Prove that the locus of the extremity of the polar subtangent of the curve

$$1/r + f(\theta) = 0$$

is $u = f'(\frac{1}{2}\pi + \theta)$.

Hence show that the locus of the extremity of the polar subtangent in the curve

$$r = (1 + \tan \tfrac{1}{2}\theta)/(m + n \tan \tfrac{1}{2}\theta)$$

is a cardioid.

49. Show that the pedal equation of the curve

$$y^2(3a - x) = (x - a)^3$$

is $p^2 = 9a^2(r^2 - a^2)/(r^2 + 15a^2)$.

50. Show that there are just two real lines which are both tangent and normal to the curve $y^2 = x^3$, and that the abscissæ of the points where either of them touches the curve and cuts it at right angles are $\frac{8}{9}$ and $\frac{2}{9}$ respectively.

CHAPTER VIII

ASYMPTOTES

8·1. Asymptote. *A straight line at a finite distance from the origin to which a tangent to a curve tends, as the distance from the origin of the point of contact tends to infinity, is called an asymptote of the curve.*

The tangent to the curve $y = f(x)$ at (x, y) is

$$Y - y = \frac{dy}{dx}(X - x),$$

or
$$Y = \frac{dy}{dx}X + \left(y - x\frac{dy}{dx}\right). \quad . \quad . \quad (1)$$

Excluding for the present asymptotes parallel to the y-axis, (1) shows that, as $x \to \infty$,

$$\frac{dy}{dx} \text{ and } y - x\frac{dy}{dx}$$

must both tend to finite limits, say m and c, in order that an asymptote might exist. If this condition is satisfied, the asymptote would be

$$y = mx + c.$$

8·11. m is equal to the limit of y x as x tends to infinity. We have seen that if $y = f(x)$ has an asymptote which is not parallel to the y-axis, then

$$\lim_{x \to \infty}\left(y - x\frac{dy}{dx}\right)$$

is finite. It follows that

$$\lim_{x \to \infty}\frac{y - x\dfrac{dy}{dx}}{x} = 0,$$

i.e.,
$$\lim_{x\to\infty}\frac{y}{x}-\lim_{x\to\infty}\frac{dy}{dx}=0.$$

Therefore $\quad \lim_{x\to\infty}(y/x)=\lim_{x\to\infty}\dfrac{dy}{dx}=m,$

by the previous article.

8·12. The asymptotes of the general algebraic curve. Let the equation to the curve be

$$\begin{aligned} a_0\, y^n + a_1\, y^{n-1}\, x + a_2\, y^{n-2}\, x^2 + \ldots + a_{n-1}\, y\, x^{n-1} + a_n\, x^n \\ + b_1\, y^{n-1} + b_2\, y^{n-2}\, x + \ldots + b_{n-1}\, y\, x^{n-2} + b_n\, x^{n-1} \\ + c_2\, y^{n-2} + \ldots + \ldots = 0, \quad . \quad . \quad . \quad (1) \end{aligned}$$

or
$$x^n\,\varphi_n\left(\frac{y}{x}\right)+x^{n-1}\,\varphi_{n-1}\left(\frac{y}{x}\right)+\ldots=0, \qquad (2)$$

where $\varphi_r\,(y/x)$ is an expression of the rth degree in y/x.

Dividing by x^n, this can be written as

$$\varphi_n\left(\frac{y}{x}\right)+\frac{1}{x}\,\varphi_{n-1}\left(\frac{y}{x}\right)+\frac{1}{x^2}\,\varphi_{n-2}\left(\frac{y}{x}\right)+\ldots=0. \quad (3)$$

Excluding the case of asymptotes parallel to the y-axis (i.e., excluding the case in which $\lim_{x\to\infty}(y/x)$ is equal to ∞), (3) gives, on taking limits, the equation

$$\varphi_n\,(\mathbf{m})=0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which determines m.

Let now $\quad \lim_{x\to\infty}\left(y-x\dfrac{dy}{dx}\right)=c.$

Then, because $\lim_{x\to\infty}(dy/dx)=m$, we shall have, in general,

$$\lim_{x\to\infty}(y-mx)=c.$$

We may therefore put $\quad y-mx=c+v, \quad . \quad . \quad . \quad (5)$

where v is a function of x which $\to 0$ as $x\to\infty$.

Substituting the value of y from this in the equation of the curve, viz., in equation (2), we get

$$x^n\varphi_n\left(m+\frac{c+v}{x}\right)+x^{n-1}\,\varphi_{n-1}\left(m+\frac{c+v}{x}\right)+\ldots=0. \ (6)$$

Expanding by Taylor's Theorem,* we have

$$x^n\left\{\varphi_n(m)+\frac{c+v}{x}\varphi_n'(m)+\frac{(c+v)^2}{2!\,x^2}\varphi_n''(m)+\dots\right\}$$
$$+x^{n-1}\left\{\varphi_{n-1}(m)+\frac{c+v}{x}\varphi_{n-1}'(m)+\dots\right\}$$
$$+x^{n-2}\{\varphi_{n-2}(m)+\dots\}+\dots=0 \quad . \quad . \quad . \quad (7)$$

On account of (4) we can omit $\varphi_n(m)$ in (7). Then, on dividing by x^{n-1} and taking limits as $x\to\infty$, we get

$$c\varphi_n'(m)+\varphi_{n-1}(m)=0, \quad . \quad . \quad . \quad (8)$$

which determines c when m has been found from (4).

Hence the asymptotes are $y=mx+c$, where m is a root of (4) and the corresponding value of c is obtained from (8).

8·13. An Easy Rule of finding the asymptotes. If, instead of $mx+c+v$, we substitute only $mx+c$ for y in (2) of the last article and expand, we get

$$x^n\left\{\varphi_n(m)+\frac{c}{x}\varphi_n'(m)+\frac{c^2}{2!\,x^2}\varphi_n''(m)+\dots\right\}$$
$$+x^{n-1}\left\{\varphi_{n-1}(m)+\frac{c}{x}\varphi_{n-1}'(m)+\dots\right\}+\dots=0. \qquad (1)$$

If we now equate to zero the coefficients of the two highest powers of x, we get precisely the equations (4) and (8) of the last article. Hence we have the following rule for determining the asymptotes:

Substitute $mx+c$ for y in the equation of the curve and equate to zero the coefficients of the two highest powers of x. Determine m and c from these. If m_1, c_1; m_2, c_2; ... are the values of m and c thus obtained, the asymptotes are

$$y=m_1x+c_1;\ y=m_2x+c_2;\ \dots$$

Ex. Find the asymptotes of

$$y^3+x^2y-2xy^2+2x^3-7xy+3y^2+2x^2+2x+2y+1=0.$$

*As the functions to be expanded are algebraic and rational, the expansions will all be finite and will always be possible (§ 6·91).

Putting $y = mx + c$, we have

$$(mx + c)^3 - x^2(mx + c) - 2x(mx + c)^2 + 2x^3 - 7x(mx + c) + 3(mx + c)^2 + 2x^2 + 2x + 2y + 1 = 0,$$

or

$$x^3(m^3 - m - 2m^2 + 2) + x^2(3m^2c - c - 4mc - 7m + 3m^2 + 2) + \dots = 0.$$

Therefore m and c are given by

$$m^3 - 2m^2 - m + 2 = 0,$$

and

$$c(3m^2 - 4m - 1) + 3m^2 - 7m + 2 = 0.$$

The first equation gives $(m - 1)(m + 1)(m - 2) = 0$.

Therefore $\qquad m = 1, -1, \text{ or } 2.$

The second equation gives,

$$c = -\frac{3m^2 - 7m + 2}{3m^2 - 4m - 1}.$$

Therefore, when $m = 1$, $\qquad c = -1$,

when $m = -1$, $\qquad c = -2$,

and when $m = 2$, $\qquad c = 0$.

Hence the asymptotes are $\qquad y = x - 1$,

$y = -x - 2$,

and $y = 2x$.

8·14. Shorter Method. We notice that $\phi_n(m)$ can be obtained at once by putting $x = 1$ and $y = m$ in the highest degree terms of the equation to the curve. Similarly $\phi_{n-1}(m)$ can be obtained by putting $x = 1$ and $y = m$ in the $(n - 1)$th degree terms. Hence we get the asymptotes more quickly as follows:

In the highest degree terms put $x = 1$ and $y = m$, thus getting $\phi_n(m)$. Equate this to zero and solve for m. Let the roots be $m_1, m_2, \dots m_n$. Next form $\phi_{n-1}(m)$ in a similar way from the terms of degree $n - 1$. Then the values of c, say $c_1, c_2, \dots c_n$, are found by substituting $m_1, m_2, \dots m_n$ in turn in the formula

$$c = -\frac{\phi_{n-1}(m)}{\phi_n'(m)}.$$

The asymptotes then are $y = m_1x + c_1, y = m_2x + c_2, \dots y = m_nx + c_n$.

Ex Solve the example of the last article by this method.

Putting $x = 1$ and $y = m$ in the 3rd degree terms, and equating to zero, we get

$$m^3 - 2m^2 - m + 2 = 0,$$

i.e., $\qquad (m - 1)(m + 1)(m - 2) = 0$, whence $m = 1, -1$, or 2.

Then $c = -\dfrac{\phi_{n-1}(m)}{\phi_n'(m)} = -\dfrac{3m^2 - 7m + 2}{3m^2 - 4m - 1} = -1, -2, 0$ respectively.

Therefore the asymptotes are $y = x - 1$, $y = -x - 2$, and $y = 2x$.

EXAMPLES

Find the asymptotes of the following curves:

1. $x^3 + 2x^2y - xy^2 - 2y^3 + xy - y^2 - 1.$
2. $x^3 + 2x^2y - xy^2 - 2y^3 + 3xy + 3y^2 + x + 1 = 0.$
3. $2x^3 + 3x^2y - 3xy^2 - 2y^3 + 3x^2 - 3y^2 + y = 3.$
4. $2x^3 - 5x^2y + 4xy^2 - y^3 + 3x^2 - 6xy + 3y^2 - 1 = 0.$
5. $4x^3 - x^2y - 4xy^2 + y^3 + 3x^2 + 2xy - y^2 = 7.$

8·2. Asymptotes might not exist. If one or more values of m, found from $\phi_n(m) = 0$, make $\phi_n'(m)$ zero, but do not make $\phi_{n-1}(m)$ zero, the equation for determining the corresponding value of c becomes

$$0.c + \phi_{n-1}(m) = 0.$$

There is no value of c which will satisfy this equation. This means that the corresponding asymptote does not exist. (The student must not think that $c = \infty$ will do. See § 1·2).

Ex. Find the asymptotes of the curve $y^2 = x$.

Putting $y = mx + c$, we get $(mx + c)^2 - x = 0$.

Equating to zero the coefficients of x^2 and x, we have

$$m^2 = 0, \text{ and } 2mc - 1 = 0.$$

The first gives $m = 0$. Then the second reduces to $-1 = 0$, which is impossible. Hence $y^2 = x$ has no asymptotes.

[If we find the equation of the tangent at (x, y) and examine how it behaves as $x \to \infty$, it is easy to see that the tangent goes off continually further and further as $x \to \infty$, so that there is no asymptote in accordance with our definition.]

8·21. Two Parallel Asymptotes. If any value of m found from $\varphi_n(m) = 0$ makes $\varphi_n'(m)$ equal to zero and also $\varphi_{n-1}(m)$ equal to zero*, the equation from which c is usually determined degenerates into the identity $0.c + 0 = 0$. To determine c we have now the equation

$$\tfrac{1}{2} c^2 \varphi_n''(m) + c\varphi_{n-1}'(m) + \varphi_{n-2}(m) = 0, \quad . \quad . \quad (1)$$

which follows from equation (7) of § 8·12, on dividing by x^{n-2} and taking limits.

Hence, corresponding to one value of m there are

*This will happen only when $\phi_n(m) = 0$ has multiple roots. See § 4·5.

now two values of c. That is, we shall get a pair of parallel asymptotes.

Reference to equation (1) of § 8·13 shows that the coefficient of x^{n-2} in that equation is the same as the left hand side of (1) above.

Hence the rule for finding asymptotes can be extended as follows: *Substitute $mx + c$ for y in the equation of the curve and equate to zero the coefficient of the highest power of x. If any value of m derived from this equation makes the coefficient of x^{n-1} identically zero, then the corresponding values of c should be determined from the equation obtained by putting the coefficient of x^{n-2} equal to zero.*

Ex. Find the asymptotes of $y^3 + x^2y + 2xy^2 - y + 1 = 0$.

Putting $y = mx + c$, we have

$$(mx + c)^3 + x^2(mx + c) + 2x(mx + c)^2 - (mx + c) + 1 = 0,$$

or

$$x^3(m^3 + m + 2m^2) + x^2(3m^2c + c + 4mc) + x(3mc^2 + 2c^2 - m) + \dots = 0.$$

Therefore m and c must satisfy

$$m^3 + 2m^2 + m = 0,$$

and

$$(3m^2 + 4m + 1)c = 0.$$

[These could also have been written down at once by § 8·14.]

The first equation gives $m = 0, -1, -1$.

The second equation then gives $c = 0$ when $m = 0$; but when $m = -1$, the second equation reduces to the identity $0.c = 0$.

Equating, therefore, the coefficient of the next lower power of x to zero, [or by finding $\phi_n''(m)$ and $\phi_{n-2}(m)$] we have

$$(3m + 2)c^2 - m = 0,$$

which gives $c = \pm 1$ after putting $m = -1$.

Hence the asymptotes are $y = 0$,

$$y = -x + 1,$$

and $y = -x - 1$.

NOTE. *The student must guard against concluding from equations like $(3m^2 + 4m + 1)c = 0$ that $c = 0$. This inference would be correct only when the coefficient of c is not zero.*

EXAMPLES

Find the asymptotes of the following curves :

1. $x^3 + x^2y - xy^2 - y^3 - 3x - y - 1 = 0.$
2. $x^3 + x^2y - xy^2 - y^3 + x^2 - y^2 - 2.$
3. $4x^3 - 3xy^2 - y^3 + 2x^2 - xy - y^2 = 1.$
4. $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0.$
5. $x^3 - 5x^2y + 8xy^2 - 4y^3 + x^2 - 3xy + 2y^2 - 1 = 0.$

8·22. More than two Parallel Asymptotes. If the coefficient of x^{n-2} equated to zero becomes

$$0.c^2 + 0.c + 0 = 0$$

for a value of m derived from $\phi_n(m) = 0$, an argument similar to that of the last article will show that c must be determined from the coefficient of the next highest power of x equated to zero; if this also fails to determine c, then the coefficient of the next highest power of x must be equated to zero and so on.

8·23. Asymptotes parallel to the x-axis. While asymptotes parallel to the x-axis would get determined when we substitute $y = mx + c$ and equate to zero the coefficients of x^n and x^{n-1}, it is usual to determine these separately first, as the method of finding them is extremely simple.

The general equation of the curve of degree n (eqn. (1) of § 8·12), arranged according to powers of x, is

$$a_n x^n + (a_{n-1}y + b_n)\, x^{n-1}$$
$$+ (a_{n-2}y^2 + b_{n-1}y + c_n)\, x^{n-2} + \ldots = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Putting $y = mx + c$, and equating to zero the coefficient of x^n (or at once by § 8·14), we get for determining m the equation

$$a_n + a_{n-1}\, m + a_{n-2}\, m^2 + \ldots = 0. \quad . \quad (2)$$

This shows that $m = 0$ will be a root only if $a_n = 0$.

Suppose then that $a_n = 0$, and consider the root $m = 0$ of (2). The corresponding asymptote is

$$y = c, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

where c is determined by the equation obtained on substituting $y = 0.x + c$ (i.e., $y = c$) in (1) and equating to

zero the coefficient of x^{n-1}. Hence c in (3) must be determined by

$$a_{n-1}c + b_n = 0. \qquad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

Now, to substitute the value of c from (4) in (3) is the same as eliminating c between (4) and (3). Hence the asymptote is

$$a_{n-1}y + b_n = 0, \qquad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which could have been obtained by equating to zero the coefficient of x^{-1} in (1).

If a_{n-1} and b_n are both zero, so that (4) becomes an identity and there are no terms involving x^{n-1} in the equation to the curve, arguments similar to those of §§ 8·21 and 8·22 will apply. We shall find that in every case *the asymptote or asymptotes parallel to the axis of x can be obtained by equating to zero the coefficient of the highest power of x, provided this is not merely a constant.*

Ex. 1. Find the asymptote, parallel to the axis of x, of the curve

$$y^3 + x^2y + 2xy^2 - y + 1 = 0$$

Here the highest power of x is x^2 and its coefficient is y. So the required asymptote is

$$y = 0. \quad \text{(Cf. the Ex. of § 8 21.)}$$

Ex. 2. Find the asymptotes, parallel to the axis of x, of the curve

$$y^4 + x^2y^2 + 2xy^2 - 4x^2 - y + 1 = 0$$

Here the coefficient of the highest power of x is $y^2 - 4$

So the required asymptotes are $y = \pm 2$.

8·24. Asymptotes parallel to the y-axis. As the form $x = c$ is not included in $y = mx + c$, the preceding methods will fail to determine asymptotes of this form. But they can be easily found from the rule given below, whose truth becomes apparent when we consider what would be the result of interchanging the axes of x and y.

The asymptotes parallel to the axis of y are obtained by equating to zero the coefficient of the highest power of y, provided this is not merely a constant.

Ex. Find those asymptotes of the curve

$$x^4 + x^2y^2 + a^2(a^2 + y^2) = 0$$

which are parallel to the y-axis.

The coefficient of y^2 equated to zero gives $x^2 - a^2 = 0$. Hence the required asymptotes are

$$x = a, \; x = -a.$$

Examples

Write down the equations of those asymptotes of the following curves which are parallel to either of the axes:

1. $x^3 \quad 3xy^2 \quad y^2 \quad 2x + y = 0.$
2. $xy^2 \quad y^3 - x - y - 1.$
3. $x^2y^3 + x^3y^2 = x^3 + y^3.$
4. $a^2x^2 - (x-1)^2(y^2 + a^2) - 0.$
5. $(x^3 + a^3)y - bx^3$

8·3. The Curve approaches the asymptote.

The distance of a point on any branch of a curve from the corresponding asymptote tends to zero as the distance of the point from the origin tends to infinity.

We have seen (§ 8·21) that the coordinates of a point on the curve $x^n\varphi_n(y/x) + x^{n-1}\varphi_{n-1}(y/x) + \ldots = 0$ satisfy the equation

$$y = mx + c + v,$$

where $y = mx + c$ is an asymptote, and v is a function of x which $\to 0$ as $x \to \infty$. So the difference of the ordinates of the curve and the asymptote for a given value of x is v. Since this $\to 0$ as $x \to \infty$, the proposition enunciated above is obvious.

Supposing the axes to be interchanged for an instant, we see that the proposition must be true for asymptotes of the form $x = c$ also.

8·31. The asymptote of the curve $y = mx + c + A/x + B/x^2 + \ldots$ Let the equation to a curve be given by the equation

$$y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \cdots, \qquad \ldots \quad (1)$$

where the series $A/x + \dots$ is convergent for sufficiently large values of x. Differentiating, we have

$$\frac{dy}{dx} = m - \frac{A}{x^2} - \dots$$

The tangent at (x, y) is therefore

$$Y - y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)(X - x),$$

or $$Y = \left(m - \frac{A}{x^2} - \frac{2B}{x^3} - \dots\right)X + c + \frac{2A}{x} + \frac{3B}{x^2} + \dots \quad (2)$$

Let x now tend to infinity. The equation (2) then tends to the equation

$$Y = mX + c.$$

We have proved, therefore, that *the asymptote of*

$$\mathbf{y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots}$$

is $$\mathbf{y = mx + c.}$$

This method is sometimes useful.

Ex. 1. Find the asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$

Here

$$\frac{y}{b} = \sqrt{\left(\frac{x^2}{a^2} - 1\right)} = \frac{x}{a}\sqrt{\left(1 - \frac{a^2}{x^2}\right)}$$

$$= \pm\frac{x}{a}\left(1 - \frac{1}{2}\frac{a^2}{x^2} - \frac{1}{8}\frac{a^4}{x^4} + \dots\right).$$

Hence the asymptotes are

$$y = bx/a.$$

and $$y = -bx/a.$$

8·32. Position of the curve with respect to the asymptote. If we can throw the equation of a given curve in form $y = mx + c + Ax^{-1} + \dots$, we can get useful information regarding the position of the curve with respect to the asymptote. The following cases arise:

I. $A \neq 0$.

Let y_1 be the ordinate of the curve and y_2 that of the asymptote when the abscissa is x_1. Then

$$y_1 = mx_1 + c + A x_1^{-1} + Bx_1^{-2} + \dots$$

and $$y_2 = mx_1 + c.$$

Now by taking x_1 sufficiently large, we can (in general) make $Bx_1^{-1} + Cx_1^{-2} + \dots$ numerically as small as we please. Let x_1 be so large that this is numerically less than A. Then

$$A + Bx_1^{-1} + Cx_1^{-2} + \dots$$

has the same sign as A; i.e., it is positive if A is positive and negative if A is negative.

Now $$y_1 - y_2 = x_1^{-1}(A + Bx_1^{-1} + \dots).$$

Therefore, if x_1 and A are of the same sign, then

$$y_1 > y_2.$$

i.e., the curve is above the asymptote. On the other hand if x_1 and A are of opposite signs, the curve lies below the asymptote.

We notice that the curve lies on opposite sides of the asymptote at the opposite ends.

II. $A = 0$.

Here the curve lies on the same side of the asymptote at opposite ends—above it, if B is positive, and below it if B is negative.

III. If B is also zero, but not C, arguments similar to those of case I would apply; and so on.

Such considerations would be found very helpful in curve-tracing (Chapter X).

8·33. Total number of asymptotes. As the equation for determining m, viz., $\varphi_n(m) = 0$, is of degree n, and so has n roots, and as one value of m gives in general one value of c (§ 8·12), it is evident that *a curve of degree n has in general n asymptotes.*

(i) If some of the roots of $\phi_n(m) = 0$ are imaginary, the corresponding asymptotes are said to be imaginary. Thus the circle $x^2 + y^2 = a^2$ has imaginary asymptotes.

(ii) There might be no asymptote corresponding to even a real root (§ 8·2). Thus the parabola $y^2 = 4ax$ has no asymptotes, even though the roots of $\phi_n(m) = 0$ are real.

(iii) There can never be more than n asymptotes, because when the equation for determining c is a quadratic (§ 8·21), $\phi_n'(m) = 0$, and so $\phi_n(m) = 0$ has a double root (§ 4·5). Therefore the two values of c correspond to the two equal roots and there would be at the most $n - 2$ other asymptotes corresponding to the remaining $n - 2$ roots. If the equation for determining c is a cubic, it can be easily shown that $\phi_n(m) = 0$ has three equal roots; and so on.

Hence we see that *a curve of degree n can never have more than n asymptotes.*

Ex. Find all the asymptotes of $x^2y^2 = a^2(x^2 + y^2)$.

The asymptotes parallel to the y-axis are $x = \pm a$; and those parallel to the x-axis are $y = \pm a$. As the equation is of the 4th degree, there cannot be more than four asymptotes. Hence all the asymptotes are

$$x = \pm a, \; y = \pm a.$$

8·34. Asymptotes by Inspection. *If the equation to a curve be of the form $F_n + P = 0$, where P is of degree $n-2$, or lower, and if $F_n = 0$ can be broken up into n linear factors which represent n straight lines no two of which are parallel or coincident, then all the asymptotes are given by $F_n = 0$.*

The supposition that $F_n = 0$ breaks up into factors which represent straight lines no two of which are parallel implies that $\varphi_n(m) = 0$ has no repeated roots. Hence the values of c do not depend on terms of lower degree than $n - 2$ (see § 8·21).

So *the asymptotes of $F_n + P = 0$ would be the same as those of $F_n = 0$*. But $F_n = 0$ is a system of n straight lines, each one of which is its own asymptote (as is obvious from the definitions of asymptote and tangent). Hence the proposition must be true.

This enables us to write down the asymptotes by mere inspection in many cases.

Ex. Find the asymptotes of the hyperbola $x^2/a^2 - y^2/b^2 = 1$. By the above proposition the asymptotes must be $x^2/a^2 - y^2/b^2 = 0$,

i.e., $$\frac{x}{a} = \pm \frac{y}{b}.$$

Examples

Find all the asymptotes of the following curves:

1. $y^3 = x^2(x - a)$.
2. $y^3 - xy^2 - x^2y + x^3 + x^2 - y^2 = 1$. [*Aligarh*, 1930]
3. $y^2(x^2 - a^2) = x$. [*Calcutta*, 1938]
4. $y(x - y)^3 = y(x - y) + 2$. [*Patna*, 1935]
5. $y^2(x - b) = x^3 + a^3$. [*Agra*, 1935]
6. $(a + x)^2(b^2 + x^2) = x^2y^2$. [*Nagpur*, 1928]
7. $2x(y - 3)^2 = 3y(x - 1)^2$. [*Agra*, 1931]

8. (i) $x^4 - y^4 = a^2xy$. (ii) $xy^2 = 4a^2(2a - x)$. [*Nagpur*, 1931]

9. $(x^2 - a^2)y^2 = x^2(x^2 - 4a^2)$. [*Agra*, 1933]

10. $a^2/x^2 + b^2/y^2 = 1$. [*Allahabad*, 1921]

11. $(x^2 - y^2)(y^2 - 4x^2) - 6x^3 + 5x^2y + 3xy^2 - 2y^3 - x^2 + 3xy - 1 = 0$. [*Nagpur*, 1935]

12. $y^3 - 2y^2x - yx^2 + 2x^3 + y^2 - 6xy + 5x^2 - 2y + 2x + 1 = 0$.

13. $(x^2 - y^2)(x + 2y + 1) + x + y + 1 = 0$.

14. $x(y - 3)^3 - 4y(x - 1)^3$.

15. $(x^2 - 3x + 2)(x + y - 2) + 1 = 0$.

16. (i) $(\alpha_1 x + \beta_1 y + \gamma_1)(\alpha_2 x + \beta_2 y + \gamma_2) + \gamma_3 = 0$.

(ii) $x^3 - y^3 = (x - 1)^2$.

(iii) $(y - a)^2(x^2 - a^2) = x^4 + a^4$. [*Lucknow*, 1933]

17. $y^2(x - 2a) = x^3 - a^3$.

18. $y^3 = x^2(2a - x)$.

19. Find the asymptotes of

$$(y - x)^2 x - 3y(y - x) + 2x = 0,$$

and examine how the curve is placed with reference to them. [*Annamalai*, 1936]

20. Prove that the curve $y = x(x^2 + a^2)/(x^2 - a^2)$ lies above the asymptote for large positive values of x.

21. Prove that the curve $x^3 - x^2y + x^2 - 2x - 1$ lies below the asymptote in the first quadrant when x is large. What is the position of the curve with respect to the asymptote for large negative values of x?

22. On which side of its oblique asymptote does the curve $y(a_0x^2 + a_1x + a_2) = b_0x^3 + b_1x^2 + b_2x + b_3$ lie?

8·35. Intersections of the Curve and its Asymptotes. A straight line cuts a curve of degree n in general in n points. As one of these points of intersection is kept fixed, and another point of intersection is made to tend to it, the straight line tends to the tangent at the former point. Hence a tangent (and therefore, as a particular case, an asymptote) will in general cut the curve in $n - 2$ points. The n asymptotes will therefore cut the curve in $n(n - 2)$ points.

Let the asymptotes of a curve of degree n be

$$F_n \equiv (y - m_1x - c_1)(y - m_2x - c_2) \dots (y - m_nx - c_n) = 0.$$

Let now the equation to any curve having asymptotes $F_n = 0$ be put in the form
$$F_n + P = 0. \qquad \dots \qquad (1)$$

Then P *cannot be of a degree higher than* $n - 2$, otherwise the terms of degree $n - 1$ and higher will be different in F_n and $F_n + P$, and so the usual method of finding asymptotes (§ 8·13 or 8·14) will give values of m, or c, or both, different from m_1, c_1; m_2, c_2; ...

Now if $S = 0$ and $S' = 0$ represent two curves, $S - \lambda S' = 0$ represents some curve through the intersections of $S = 0$ and $S' = 0$.

Therefore $F_n + P - F_n = 0$, i.e. $P = 0$, is some curve passing through the intersections of $F_n + P = 0$ and $F_n = 0$. So, as P is of a degree not higher than $n - 2$, we get the following proposition:

A curve of degree $n - 2$, or lesser, can be made to pass through the $n(n - 2)$ points of intersection of a curve of degree n and its asymptotes.

NOTE. Of course, many other curves can be made to pass through these points, but they are generally of a higher degree than $n - 2$.

Ex. Show that the intersection of the curve

$$2y^3 - 2x^2y - 4xy^2 + 4x^3 - 14xy + 6y^2 - 4x^2 + 6y + 1 = 0$$

and its asymptotes lie on the straight line $8x + 2y + 1 = 0$.

The asymptotes of the curve are, as in the example of § 8·13,

$$(y - x + 1)(y + x + 2)(y - 2x) = 0,$$

or, $y^3 - x^2y - 2xy^2 + 2x^3 - 7xy + 3y^2 + 2x^2 + 2y - 4x = 0.$

Multiplying this by 2 and subtracting from the equation of the curve, we get

$$8x + 2y + 1 = 0,$$

on whch the points of intersection must lie.

EXAMPLES

1. Show that the asymptotes of the cubic

$$x^3 - 2y^3 + xy(2x - y) + y(x - y) + 1 = 0$$

cut the curve in three points which lie on the straight line

$$x - y + 1 = 0.$$

2. Find the equation of the straight line on which lie the three points of intersection of the curve

$$(x + a)y^2 = (y + b)x^2$$

and its asymptotes.

3. Find the asymptotes of the curve

$$x^2(x + y)(x - y)^2 + ax^3(x - y) - a^2y^3 = 0.$$

Form the equation of the quartic curve which has $x = 0$, $y = 0$, $y = x$, $y = -x$, for asymptotes, which passes through the point (a, b), and which cuts its asymptotes again in eight points lying upon the circle $x^2 + y^2 = a^2$. [*Lucknow*, 1932]

4. Show that the eight points of intersection of the curve

$$xy(x^2 - y^2) + x^2 + y^2 = a^2$$

and its asymptotes lie on a circle whose centre is at the origin.

5. Show that the eight points of intersection of the curve

$$x^4 - 5xy^2 + 4y^4 + x^2 - y^2 + x + y + 1 = 0$$

and its asymptotes lie on a rectangular hyperbola.

8·36. Curvilinear Asymptotes. If the equation to a curve can be thrown in the form

$$y = a_0x^n + a_1 x^{n-1} + \dots + a_n + \frac{A}{x} + \frac{B}{x^2} + \dots, \qquad (1)$$

then $\qquad y = a_0 x^n + a_1 x^{n-1} + \dots + a_n \qquad \dots\dots (2)$

is said to be asymptotic to (1).

In particular, if the curve can be put as

$$y = ax^2 + bx + c + \frac{A}{x} + \frac{B}{x^2} + \dots, \quad a \neq 0,$$

it possesses the *parabolic asymptote*

$$y = ax^2 + bx + c.$$

It will be observed that if two curves are asymptotic in accordance with the above definition, the difference of their ordinates for a given value of x tends to zero as $x \to \infty$. (cf. § 8·3).

Similar definitions hold for curves in which x can be expanded in negative powers of y.

Ex. Show that the curve

$$ax^2 y = x^4 + a^2x^2 + a^4$$

has a parabolic asymptote.

Here $$y = \frac{x^2}{a} + a + \frac{a^3}{x^2}.$$

Hence the given curve has the parabolic asymptote

$$ay = x^2 + a^2.$$

EXAMPLES

1. Show that the curve represented by $x^3 + aby - axy = 0$ has a parabolic asymptote $x^2 + bx + b^2 = ay$.

2. Find the rectilinear and parabolic asymptotes of the curve

$$x = (y^3 - a^3)/ay. \qquad \text{[\textit{Patna}, 1932]}$$

8·4. Asymptotes to non-algebraic curves. The method of substituting $y = mx + c$ and equating to zero the coefficients of the two highest powers of x applies only to algebraic curves. In the case of non-algebraic curves the asymptotes can be found in simple cases by applying the definition, or by the expansion of y in negative powers of x.

Ex. Find the asymptotes of $y = \tan x$.

Here $$\frac{dy}{dx} = \sec^2 x.$$

Hence the tangent at (x, y) is

$$Y - \tan x = (X - x)\sec^2 x,$$

or $$Y\cos^2 x - \sin x \cos x = X - x. \quad . \quad . \quad . \quad (1)$$

Now as $x \to \frac{1}{2}\pi$ from the left, $y \to \infty$, and the distance of (x, y) from the origin tends to infinity. Hence, taking the limit of (1) as $x \to \frac{1}{2}\pi$, the corresponding equation is

$$Y.0 - 0 = X - \tfrac{1}{2}\pi,$$

i.e., $$X = \tfrac{1}{2}\pi.$$

This is one asymptote. The other asymptotes are $X = -\frac{1}{2}\pi, \pm \frac{3}{2}\pi, \ldots$

Examples

Find the asymptotes of

1. $y = e^{-x^2}$.
2. $y = e^{ax}$.
3. $y = \sec x$.
4. $y = \operatorname{cosec} x$.

8·5. Polar Curves. *If* α *be a root of the equation* $f(\theta) = 0$, *then*

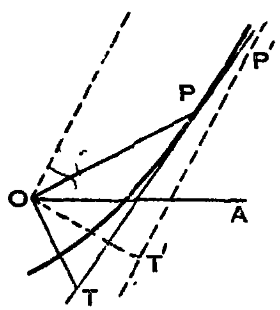
$$r \sin(\theta - \alpha) = 1/f'(\alpha)$$

is an asymptote of the curve

$$1/r = f(\theta).$$

Let P be any point (r, θ) on the curve

$$1/r = f(\theta). \quad . \quad . \quad (1)$$



Let OT be the perpendicular to the radius vector OP, cutting the tangent at P in T. Then, OT is the polar subtangent and so

$$OT = r^2 \frac{d\theta}{dr}.$$

Now let $\theta \to \alpha$.

Then $f(\theta)$ will tend to 0, because $f(\alpha) = 0$ by hypothesis, and we suppose $f(\theta)$ to be a continuous function.

Therefore r will $\to \infty$ by equation (1), values of θ on one side of α alone being considered.

Thus PT will tend to the asymptote, and OT will tend to the value of $r^2\, d\theta/dr$ at $\theta = \alpha$.

Also OP and PT will tend to become parallel, and thus the angle OTP will tend to a right angle (as shown by the dotted lines in the figure).

We see, therefore, that the asymptote is the straight line which is parallel to the radius vector $\theta = \alpha$, and is situated at a distance from O equal to

$$\left(r^2 \frac{d\theta}{dr}\right)_{\theta=\alpha}$$

to the right of an observer looking from the origin in the direction $\theta = \alpha$.

So the perpendicular on the asymptote, viz., OT', makes an angle of $\frac{1}{2}\pi$ with the radius vector $\theta = \alpha$, and therefore an angle of $\alpha - \frac{1}{2}\pi$ with the initial line. But the polar equation of the straight line the perpendicular on which from the origin makes an angle β with the initial line and is of length p is

$$r \cos(\theta - \beta) = p.$$

Hence the equation of the asymptote must be

$$r \cos(\theta - \alpha + \tfrac{1}{2}\pi) = \left(r^2 \frac{d\theta}{dr}\right)_{\theta=\alpha}.$$

Substituting the value of $r^2\, d\theta/dr$ from the equation of the curve, the asymptote becomes

$$\mathbf{r \sin(\theta - \alpha) = 1/f'(\alpha).}$$

Ex. Find the asymptotes of the curve $r \sin n\theta = a$.

The equation to the curve can be written as

$$1/r = (1/a) \sin n\theta = f(\theta), \text{ say.}$$

Now $\quad f(\theta) = 0$ if $n\theta = m\pi$, where m is an integer.

Therefore $\quad \theta = m\pi/n = \alpha$, say.

Also $\quad f'(\theta) = (1/a).\, n.\cos n\theta.$

Therefore $\quad f'(\alpha) = (1/a).\, n.\cos m\pi.$

The asymptotes, therefore, are

$$r \sin\left(\theta - \frac{m\pi}{n}\right) = \frac{a}{n \cos m\pi}.$$

8·6. Circular asymptotes. If the equation to a curve is

$$r = f(\theta),$$

and if

$$\lim_{\theta \to \infty} f(\theta) = a,$$

then the circle $r = a$ is called *a circular asymptote* of the curve $r = f(\theta)$. It is evident that the curve approaches the circle as $\theta \to \infty$.

Ex. Find the circular asymptote of the curve

$$r = \frac{\theta^3 - 1}{\theta^3 + 1}$$

Dividing by θ^3, and taking limits, we see that the circular asymptote is $r = 1$.

EXAMPLES

Find the rectlinear asymptotes of the following curves:

1. $r = 2/(1 + 2 \sin \theta)$.
2. $r\theta = a$. [*Alig.*, '3[illegible]]
3. $r = 4(\sec \theta + \tan \theta)$.
4. $r \cos \theta = 4 \sin^2 \theta$.
5. $r = 2\theta/\sin \theta$.
6. $r \sin \theta = 2 \cos 2\theta$.
7. $r = \pi/\theta$.
8. $r \cos \theta = a \sin \theta$.
9. $r \sin 2\theta = a$.
10. $r = a \operatorname{cosec} \theta + b$.

Find the circular asymptotes of the following curves:

11. $r(\theta^2 + 1) = \theta^2 - 1$.
12. $r = (3\theta^2 + 2\theta + 1)/(\theta^2 + \theta + 1)$.

EXAMPLES ON CHAPTER VIII

Find the asymptotes of the following curves:

1. $x^2y^2 - x^2y - xy^2 + x + y + 1 = 0$. [*Calcutta*, 1937]
2. $x^2y + xy^2 + xy + y^2 + 3x = 0$. [*Punjab*, 1936]
3. $(x + y)^2 (x + 2y + 2) = x + 9y + 2$. [*Delhi*, 1935]
4. $2x(y - 3)^2 = 3y(x - 1)^2$. [*Allahabad*, 1923]
5. $x^2 (x - y)^2 + a^2 (x^2 - y^2) - a^2 xy = 0$. [*Nagpur*, 1933]
6. $x^5 - y^5 = a^3 xy$.
7. $(a^2/x^2) - (b^2/y^2) = 1$. [*Allahabad*, 1926]
8. $(x - 2y)^2 (x - y) - 4y(x - 2y) - (8x + 7y) = 0$. [*Bombay*, 1936]
9. $y^3 - 5xy^2 + 8x^2y - 4x^3 - 3y^2 + 9xy - 6x^2 + 2y - 2x + 1 = 0$. [*Allahabad*, 1933]
10. $r = 4(1 + \sec 2\theta)$.

11. $r^n \sin n\theta = a^n$.

12. $r\theta \cos \theta = a \cos 2\theta$. [*Allahabad*, 1929]

13. $r(e^\theta - 1) = a(e^\theta + 1)$. Find also the circular asymptote.

14. An asymptote is sometimes defined as a straight line which cuts the curve in two points at infinity. Criticise this definition and replace it by a correct definition.

Find the asymptotes of

$$y^2 = x^4/(a^2 - x^2).$$ [*Aligarh*, 1935]

15. Show that the asymptotes of

$$x^2y^2 - a^2(x^2 + y^2) - a^3(x + y) + a^4 = 0$$

form a square, through two of whose angular points the curve passes.

16. Show that the curve

$$r = a \sec m\theta + b \tan m\theta$$

has two sets of asymptotes, members of the first set touching one fixed circle, and those of the other another fixed circle.

17. Find all the asymptotes of the curve

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

Show that the asymptotes meet the curve again in three points which lie on a straight line; and find the equation of this line.

[*Benares*, 1933]

18. Show that the four asymptotes of the curve

$$(x^2 - y^2)(y^2 - 4x^2) + 6x^3 - 5x^2y - 3xy^2 + 2y^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on the circle $x^2 + y^2 = 1$.

[*Benares*, 1937]

19. Prove that if the curve

$$x^n f_0(y/x) + x^{n-1} f_1(y/x) + \dots = 0$$

have two asymptotes parallel to $y = mx$, not situated at infinity, they will be given by $y = mx + c$ and

$$\tfrac{1}{2} c^2 f_0''(m) + c f_1'(m) + f_2(m) = 0.$$

Find the asymptotes of the curve

$$(x + y)^2 (x + 2y + 2) = x + 9y - 2.$$ [*Allahabad*, 1919]

20. Find the asymptotes of the curve whose equation is

$$(x - 3)(x - 2) y^2 - 9x^2 = 0,$$

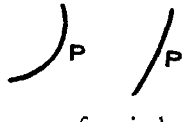
and determine on which side of each asymptote the corresponding branch of the curve lies. [*Agra*, 1929]

21. Show that there is an infinite series of parallel asymptotes to the curve $r = a/\theta \sin \theta + b$, and show that their distances from the pole are in Harmonic Progression. [*Benares*, 1935]

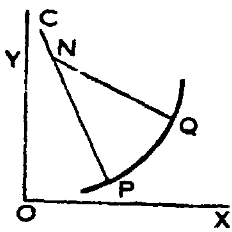
CHAPTER IX

CURVATURE

9·1. Of the two curves shown here, one bends more sharply than the other; in other words, one has a greater *curvature* than the other. But in order to get a quantitative estimate of curvature, we shall first have to give a careful definition. If P be a given point on the curve, and the part of the curve in a small neighbourhood of P be regarded roughly as an arc of a circle, we notice from the figure that the radius of such a circle would be small when the curvature is great, and vice versa. Hence we adopt the definitions given below.



9·11. Definitions. Let P be a given point on a given curve, and Q any other point on it. Let the normals at P and Q intersect in N. If N tends to a definite position C as Q tends to P, then C is called the *centre of curvature* of the curve at P.



N must tend to C whether Q tends to P from the right or from the left.

The reciprocal of the distance CP is called the *curvature* of the curve at P.

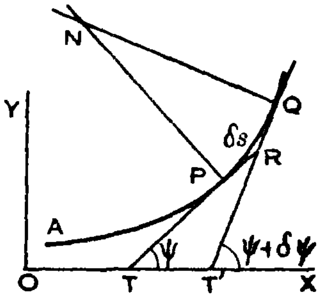
The circle with its centre at C and radius CP is called the *circle of curvature* of the curve at P.

The distance CP is called the *radius of curvature* of the curve at P. The radius of curvature is usually denoted by the Greek letter ρ.

Any chord of the circle of curvature is called a *chord of curvature.*

9·12. A formula for the radius of curvature. Let P be a given point on the curve APQ, and let Q be any other point.

Let the length of the arc AP, measured from a fixed point A on the curve, be s, and let the length from A up to Q be $s + \delta s$.



Let the tangent PT at P to the curve make with any line (which might be taken to be the axis of x) the angle ψ, and let the tangent QT' at Q make with the same line the angle $\psi + \delta\psi$. Let these tangents meet in R.

Let the normals at P and Q be PN and QN respectively, N being their point of intersection. Join PQ.

Then the radius of curvature at $P = \rho = \lim_{\delta s \to 0} PN$.

Now from the triangle PNQ,

$$\frac{PN}{\text{chord } PQ} = \frac{\sin \angle NQP}{\sin \angle PNQ} = \frac{\sin \angle NQP}{\sin \angle TRT'} = \frac{\sin \angle NQP}{\sin \delta\psi}.$$

Also as $\delta s \to 0$, $\delta\psi \to 0$.

$$\text{Hence}\quad \rho = \lim_{\delta\psi \to 0} \frac{\text{chord } PQ \sin \angle NQP}{\sin \delta\psi}$$

$$= \lim_{\delta\psi \to 0} \frac{\text{chord } PQ}{\delta s} \cdot \frac{\delta s}{\delta\psi} \cdot \frac{\delta\psi}{\sin \delta\psi} \cdot \sin \angle NQP.$$

But as $\delta\psi \to 0$, $\text{chord } PQ/\delta s \to 1$, by § 7·51,

$\delta\psi/\sin \delta\psi \to 1$, as is well known,

$\angle NQP \to \frac{1}{2}\pi$,

because $\angle NQP = \frac{1}{2}\pi - \angle PQT'$ and $\angle PQT' \to 0$ (§ 7·5),

and $\delta s/\delta\psi \to ds/d\psi$ (§ 4·2).

$$\text{Hence}\qquad \rho = \frac{ds}{d\psi}.$$

The relation between s and ψ for any curve is called the *intrinsic equation* of the curve. If we consider any arc PQ, the angle between the tangents to the curve at P and Q is called *the angle of contingence* of the arc PQ.

The above proof shows that we might say that at P

$$\rho = \lim_{PQ \to 0} \left(\frac{\text{arc } PQ}{\text{the angle of contingence of the arc } PQ} \right).$$

Ex. Find ρ for the catenary whose intrinsic equation is

$$s = c \tan \psi.$$

We have $\rho = ds/d\psi = c \sec^2 \psi$.

9·13. Cartesian Formula for Radius of Curvature.

We know that

$$\frac{dy}{dx} = \tan \psi$$

Therefore

$$\frac{d^2y}{dx^2} = \sec^2 \psi \cdot \frac{d\psi}{dx}$$

$$= \sec^2 \psi \cdot \frac{d\psi}{ds} \cdot \frac{ds}{dx}$$

$$= \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\} \cdot \frac{1}{\rho} \cdot \left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{1}{2}}$$

Hence

$$\rho = \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2y}{dx^2}}.$$

We have not considered the sign of ρ. It is customary to attach that sign to the radical which would give a positive sign to ρ.

The definition of the radius of curvature shows that its value depends only on the curve and not on the axes. Hence, interchanging the axes of x and y, we obtain.

$$\rho = \frac{\left\{ 1 + \left(\frac{dx}{dy}\right)^2 \right\}^{\frac{3}{2}}}{\frac{d^2x}{dy^2}},$$

which is useful when the tangent is parallel to the y-axis.

Ex. 1. Find the radius of curvature at (x, y) for the curve

$$a^2y = x^3 - a^3.$$

Here

$$a^2 \frac{dy}{dx} = 3x^2 \text{ and } a^2 \frac{d^2y}{dx^2} = 6x.$$

Hence
$$\rho = \frac{\left(1 + \frac{9x^4}{a^4}\right)^{\frac{3}{2}}}{\frac{6x}{a^2}} = \frac{(a^4 + 9x^4)^{3/2}}{6a^4x}.$$

Ex. 2. If a curve is defined by the equations $x = f(t)$ and $y = \phi(t)$, prove that the curvature $1/\rho$ is equal to
$$\frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}},$$
where accents denote differentiations with respect to t. [*Alld.*, 1933]

As
$$\frac{dy}{dx} = \frac{y'}{x'},$$
and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{y'}{x'}\right) = \left(\frac{d}{dt}\frac{y'}{x'}\right) \cdot \frac{dt}{dx}$$
$$= \frac{y'' x' - y' x''}{x'^2} \cdot \frac{1}{x'},$$
we have
$$\frac{1}{\rho} = \frac{\frac{d^2y}{dx^2}}{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}} = \frac{\frac{y'' x' - y' x''}{x'^3}}{\left\{1 + \left(\frac{y'}{x'}\right)^2\right\}^{\frac{3}{2}}} = \frac{x' y'' - y' x''}{(x'^2 + y'^2)^{3/2}}.$$

Ex. 3. Find the radius of curvature at the point t on the curve $x = a \cos t$, $y = b \sin t$.

Here
$$\frac{dx}{dt} = -a \sin t, \quad \frac{dy}{dt} = b \cos t.$$

Hence
$$\frac{dy}{dx} = -\frac{b}{a} \cot t.$$

Therefore
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right) \cdot \frac{dt}{dx}$$
$$= \frac{d}{dt}\left(-\frac{b}{a} \cot t\right) \cdot \frac{dt}{dx}$$
$$= +\frac{b}{a} \operatorname{cosec}^2 t. \left(-\frac{1}{a}\right) \operatorname{cosec} t.$$

Therefore
$$\rho = \frac{\{1 + (b/a)^2 \cot^2 t\}^{3/2}}{(b/a^2) \operatorname{cosec}^3 t} = \frac{(a^2 \sin^2 t + b^2 \cos^2 t)^{3/2}}{ab}.$$

EXAMPLES

Find the radius of curvature at the point (s, ψ) on the following curves :

1. $s = 8a \sin^2 \frac{1}{6} \psi$ (Cardioid).

2. $s = 4a \sin \psi$ (Cycloid).

3. $s = c \log \sec \psi$ (Tractrix)

Find the radius of curvature of the following curves :

4. $y = e^x$, at the point where it crosses the y-axis.

5. $\sqrt{x} + \sqrt{y} = 1$ at $(\frac{1}{4}, \frac{1}{4})$. [*Madras*, 1936]

Find the radius of curvature at the point (x, y) on the following curves :

6. $y^2 = 4ax$ 7. $ay^2 = x^3$ [*Agra*, 1933]

8. $y = a \log \sec (x/a)$

9. $x^{2/3} + y^{2/3} = a^{2/3}$. [*Delhi*, 1936]

10. Prove that for the ellipse $x^2/a^2 + y^2/b^2 = 1$,

$$\varrho = a^2b^2/p^3,$$

p being the perpendicular from the centre upon the tangent at (x, y). [*Lucknow*, 1934]

✓11. Prove that at the point $x = \frac{1}{2}\pi$ of the curve $y = 4 \sin x - \sin 2x$, $\varrho = 5\sqrt{5}/4$ [*Benares*, 1930]

12. Prove that in the curve $x = a \cos^3 \theta$, $y = a \sin^3 \theta$,

$$\varrho = 3a \sin \theta \cos \theta$$ [*Andhra*, 1937]

13. In the cycloid

$$x = a(\theta + \sin \theta), \quad y = a(1 - \cos \theta),$$

prove that $\varrho = 4a \cos \frac{1}{2}\theta$ [*Allahabad*, 1924]

14. If CP, CD be a pair of conjugate semi-diameters of an ellipse, prove that the radius of curvature at P is CD^3/ab, a and b being the lengths of the semi-axes of the ellipse [*Nagpur*, 1927]

15.✓ Prove that the radius of curvature of the catenary

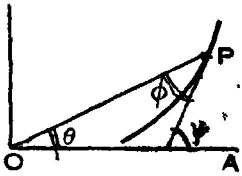
$$y = \frac{1}{2}a\,(e^{x/a} + e^{-x/a})$$

is y^2/a, and that of the catenary of uniform strength

$$y = c \log \sec (x/c)$$

is $c \sec (x/c)$. [*Aligarh*, 1932]

9·14. Formula for Pedal Equations. The relation between θ, φ and ψ (as is evident from the figure) is



$$\psi = \theta + \varphi. \qquad (1)$$

We have, therefore,

$$\frac{1}{\rho} = \frac{d\psi}{ds} = \frac{d\theta}{ds} + \frac{d\varphi}{ds} = \frac{d\theta}{ds} + \frac{d\varphi}{dr}\frac{dr}{ds}$$

$$= \frac{1}{r}\sin\varphi + \cos\varphi\frac{d\varphi}{dr} \quad \text{(by § 7·54)}$$

$$= \frac{1}{r}\left(\sin\varphi + r\cos\varphi\frac{d\varphi}{dr}\right) = \frac{1}{r}\frac{d}{dr}(r\sin\varphi) = \frac{1}{r}\frac{dp}{dr}$$

Hence $$\boldsymbol{\rho = r\frac{dr}{dp}}.$$

Ex. If the pedal equation of a curve is $p^2 = ar$, find ρ.

By differentiation, $2p = a\,dr/dp$.

Therefore $\rho = 2rp/a = 2p^3/a^2$.

9·15. Formula for Polar Equations. We know that

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

By differentiation we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^4}\cdot 2\frac{dr}{d\theta}\cdot\frac{d}{d\theta}\left(\frac{dr}{d\theta}\right)\cdot\frac{d\theta}{dr}$$

$$= -\frac{2}{r^3} - \frac{4}{r^5}\left(\frac{dr}{d\theta}\right)^2 + \frac{2}{r^4}\frac{d^2r}{d\theta^2}$$

Hence $$\rho = r\frac{dr}{dp} = \frac{r\dfrac{1}{p^3}}{\dfrac{1}{r^5}\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right\}}$$

$$= \frac{r^6\left\{\dfrac{1}{r^2} + \dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2\right\}^{3/2}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}.$$

Therefore $$\rho = \frac{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2}}.$$

Ex. Show that for the curve $r^n = a^n \cos n\theta$, the radius of curvature is $a^n r^{-n+1}/(n+1)$

By logarithmic differentiation, we have

$$\frac{n}{r}\frac{dr}{d\theta} = -\frac{n \sin n\theta}{\cos n\theta}$$

i.e., $$\frac{dr}{d\theta} = -r \tan n\theta$$

Therefore $$\frac{d^2r}{d\theta^2} = -\frac{dr}{d\theta}\tan n\theta - nr\sec^2 n\theta$$
$$= r\tan^2 n\theta - nr\sec^2 n\theta$$

Hence $$\rho = \frac{\{r^2 + r^2\tan^2 n\theta\}^{3/2}}{r^2 + 2r^2\tan^2 n\theta - r^2\tan^2 n\theta + nr^2\sec^2 n\theta}$$
$$= \frac{r^3\sec^3 n\theta}{(n+1)r^2\sec^2 n\theta} = \frac{r}{(n+1)\cos n\theta} = \frac{a^n r^{-n+1}}{n+1}$$

Examples

Find the radius of curvature at the point (p, r) on the following curves:

1. $r^3 = 2ap^2$ (Cardioid). 2. $pr = a^2$ (Hyperbola)

3. $r^3 = a^2p$ (Lemniscate) 4 $pa^n = r^{n+1}$ (Sine Spiral).

5. $p^2 = r^4/(r^2 + a^2)$ (Archimedian Spiral).

6. $\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2b^2}$ (Ellipse)

7. In the curve $p = r^{n+1}/a^n$, show that the radius of curvature varies inversely as the $(n-1)$th power of the radius vector

[*Patna*, 1935]

Find the radius of curvature at the point (r, θ) on each of the following curves:

8. $r = a/\theta$ 9. $r = a\cos\theta$

10. $r(1 + \cos\theta) = a$ 11 $r^n = a^n \sin n\theta$.

12. $r = a(2\cos\theta - 1)$. 13. $\theta = (r^2 - a^2)^{1/2}a^{-1} - \cos^{-1}(a/r)$

14. Prove that for the curve $r = a(1 + \cos\theta)$, ρ^2/r is constant

[*Patna*, 1931]

15. Show that at any point on the equiangular spiral $r = a e^{\theta \cot a}$, $\rho = r \operatorname{cosec} \alpha$, and that it subtends a right angle at the pole.

[*Allahabad*, 1926]

9·16. Miscellaneous formulae for ρ.

(i) **When x and y are given as function of s.**

$$\cos \psi = \frac{dx}{ds} \quad (\S\ 7\cdot 54).$$

Differentiating w. r. t. s, $-\sin \psi . \dfrac{1}{\rho} = \dfrac{d^2x}{ds^2}$.

Therefore $\qquad \rho = -\left(\dfrac{dy}{ds}\right) / \left(\dfrac{d^2x}{ds^2}\right)$.

(ii) Similarly, starting with $\sin \psi = dy/ds$, we get

$$\rho = \left(\frac{dx}{ds}\right) / \left(\frac{d^2y}{ds^2}\right).$$

(iii) Squaring and adding the values of $\dfrac{1}{\rho}\dfrac{dx}{ds}$ and $\dfrac{1}{\rho}\dfrac{dy}{ds}$ obtained from the above, we have

$$\frac{1}{\rho^2} = \left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2.$$

(iv) **Curvature at the origin.** Curvature at the origin can be found by substituting $x = 0$, $y = 0$ in the value of ρ obtained from the formula of § 9 13. But if y can be easily expanded in powers of x by algebraic or trigonometric methods, and

$$y = px + qx^2/2! + \dots,$$

so that the curve passes through the origin, we know that

$$p = \left(\frac{dy}{dx}\right)_{x=0,\ y=0}, \quad q = \left(\frac{d^2y}{dx^2}\right)_{x=0,\ y=0}.$$

Hence the value of ρ at the origin can be found in such cases more simply by using the formula.

$$\rho \text{ (at the origin)} = (1 + p^2)^{3/2}/q.$$

(v) **Newtonian Method.** If a curve passes through the origin and the axis of x is the tangent at the origin, we have (using dashes to denote differentiation) the following values for x, y, and y' at the origin:

$$x = 0, \ y = 0, \ y' = 0.$$

Thus, expanding by Maclaurin's Theorem,

$$y = 0 + 0.x + qx^2/2! + \dots$$

Dividing by x^2 and taking limits, we get

$$\lim_{x \to 0} (2y/x^2) = q.$$

Therefore by (iv) we have at the origin

$$\rho = \frac{(1 + p^2)^{3/2}}{q} = \frac{1}{q} = \lim_{x \to 0} \frac{x^2}{2y}.$$

(vi) Similarly, if a curve passes through the origin and the ax of y is the tangent there, the value of ρ at the origin is

$$\lim_{x \to 0} \frac{y^2}{2x}.$$

This formula and the one above are known as Newton's fo mulae.

(vii) **Polar Form : Alternative Formula.** By putting $1/u$ fc r in the formula for polar equations, and remembering that $dr/d\theta =$ $- u^{-2}\, du/d\theta$, we can show easily that

$$\rho = \frac{(u^2 + u'^2)^{3/2}}{u^3 (u + u'')}.$$

(viii) **Tangential Polar Form.** We can also derive a formu for use when the relation between p and ψ is given. We have

$$\frac{dp}{d\psi} = \frac{dp}{dr} \cdot \frac{dr}{ds} \cdot \frac{ds}{d\psi} = \frac{dp}{dr} \cdot \cos \phi \cdot r \frac{dr}{dp} \text{ (by §§ 7·54, 9·12 and 9·1}$$

$$= r \cos \phi.$$

Therefore, $$p^2 + \left(\frac{dp}{d\psi}\right)^2 = r^2 \sin^2 \phi + r^2 \cos^2 \phi = r^2.$$

Differentiating with respect to p,

$$2r \frac{dr}{dp} = 2p + 2 \frac{dp}{d\psi} \cdot \frac{d^2p}{d\psi^2} \cdot \frac{d\psi}{dp},$$

or $$\rho = p + \frac{d^2p}{d\psi^2}.$$

Ex. 1. Find the curvature at the origin of the curve

$$a_1 x + a_2 y + b_1 x^2 + b_2 xy + b_3 y^2 + c_1 x^3 + \dots = 0.$$

Substituting $px + qx^2/2! + \dots$ for y, we have the identity

$$a_1 x + a_2 (px + qx^2/2! + \dots) + b_1 x^2 + b_2 x (px + \dots)$$
$$+ b_3 (px + \dots)^2 + \dots = 0.$$

Equating to zero the coefficients of x and x^2, we get

$$a_1 + a_2 p = 0,$$

and
$$\tfrac{1}{2}a_2 q + b_1 + b_2 p + b_3 p^2 = 0.$$

Hence
$$p = -a_1/a_2\,;\quad q = -2\,\frac{b_1 + b_2 p + b_3 p^2}{a_2}.$$

Therefore
$$\rho = \frac{(1+p^2)^{3/2}}{q} = -\frac{1}{2}\,\frac{(a_1^2 + a_2^2)^{3/2}}{b_1 a_2^2 - b_2 a_1 a_2 + b_3 a_1^2}.$$

Ex. 2. Find the curvature at the origin of the curve

$$5x^3 + 7y^3 + 4x^2y + xy^2 + 2x^2 + 3xy + y^2 + 4x = 0.$$

It is easy to verify that the tangent at the origin is the axis of y. (We can infer the same at once by § 10·2.)

Dividing by x,

$$5x^2 + 7y.(y^2/x) + 4xy + y^2 + 2x + 3y + (y^2/x) + 4 = 0.$$

Taking limits as $x \to 0$,

$$\lim_{x\to 0}(y^2/x) + 4 = 0,$$

whence (numerically) $\rho = 2$.

Ex. 3. Find the radius of curvature at (r, θ) on the curve $r\theta^2 = a$.

Here
$$au = \theta^2,\quad au' = 2\theta,\quad au'' = 2.$$

Therefore
$$\rho = a\,\frac{(\theta^4 + 4\theta^2)^{3/2}}{\theta^6(\theta^2+2)} = a\,\frac{(\theta^2+4)^{3/2}}{\theta(\theta^2+2)}.$$

EXAMPLES

Find the radius of curvature for the curve in which

1. $x = c\log\{s + \sqrt{(c^2+s^2)}\},\ y = \sqrt{(c^2+s^2)}.$
2. $x = 2a\sin^{-1}(s/4a) + \tfrac{1}{2}s\sqrt{(1 - s^2/16a^2)},\ y = s^2/8a.$

Find the curvature at the origin of the following curves:

3. $y = x^4 - 4x^3 - 18x^2.$
4. $y = x^3 + 5x^2 + 6x.$
5. $y^2 - 3xy - 4x^2 + x^3 + x^4y + y^5 = 0.$
6. Show that the radii of curvature of the curve

$$y^2 = x^2(a+x)/(a-x)$$

at the origin are $\pm a\sqrt{2}$. [*Patna*, 1934]

7. Find the radius of curvature at the point (r, θ) of the curve

$$u^2 = a^{-2}\cos^2\theta + b^{-2}\sin^2\theta.$$

8. Show that for the epi-cycloid $p = a \sin b\psi$, ρ varies as p.

9. Show that the circle of curvature at the origin for the curve $x + y = ax^2 + by^2 + cx^3$ is $(a + b)(x^2 + y^2) = 2x + 2y$.

[*Bombay*, 1935]

10. Find the radii of curvature at the origin of the two branches of the curve given by the equations

$$y = t - t^3, \; x = 1 - t^2. \qquad [\textit{Punjab}, 1936]$$

9·2. Centre of Curvature. Let P be the point (x, y) and Q the point $(x + h, y + k)$. See the figure of § 9·11. Let the centre of curvature for P be (α, β).

The normal at P is

$$(Y - y)\,\varphi(x) + (X - x) = 0. \qquad \ldots (1)$$

where $\varphi(x)$ stands for dy/dx.

The normal at Q is

$$(Y - y - k)\,\varphi(x + h) + (X - x - h) = 0. \quad (2)$$

To find the ordinate of the point of intersection of these normals, subtract (1) from (2). We have

$$(Y - y)\,\{\varphi(x + h) - \varphi(x)\} - k\,\varphi(x + h) - h = 0. \quad (3)$$

Dividing by h,

$$(Y - y)\frac{\varphi(x + h) - \varphi(x)}{h} - \varphi(x + h).\frac{k}{h} - 1 = 0.$$

Now as $h \to 0$, $Y \to \beta$ by defintion,

$$\frac{\varphi(x + h) - \varphi(x)}{h} \to \frac{d}{dx}\varphi(x), \text{ i.e., } \frac{d^2y}{dx^2},$$

and

$$\frac{k}{h} \to \frac{dy}{dx}.$$

Hence we get, on taking limits,

$$(\beta - y)\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - 1 = 0,$$

i.e.,

$$\beta = y + \frac{1 + \left(\frac{dy}{dx}\right)^2}{\frac{d^2y}{dx^2}}$$

As (α, β) is a point on (1), we get also

$$\alpha = x - \frac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}}.$$

Sometimes these formulæ are derived by showing geometrically that $\alpha = x - \rho \sin \psi$, $\beta = y + \rho \cos \psi$, the curve being drawn convex towards the x-axis, when ψ is positive. If, however, a curve concave to the x-axis is taken, the result would be wrong. The trouble is that the sign of ρ has not been defined. Hence the geometrical proof should be discarded.

Ex. Find the coordinates of the centre of curvature for the point (x, y) on the parabola $y^2 = 4ax$.

Here $\qquad 2y \dfrac{dy}{dx} = 4a$, i.e., $\dfrac{dy}{dx} = \dfrac{2a}{y} = a^{1/2} x^{-1/2}$.

Therefore, $\qquad \dfrac{d^2y}{dx^2} = -\tfrac{1}{2} a^{1/2} x^{-3/2}$.

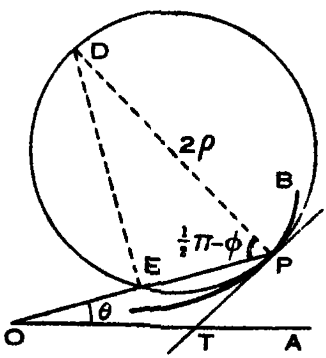
Hence $\qquad \alpha = x - \dfrac{2a^{1/2} x^{-1/2} (1 + ax^{-1})}{a^{1/2} x^{-3/2}}$

$\qquad = 2a + 3x.$

Again $\qquad \beta = 2a^{1/2} x^{1/2} - \dfrac{1 + ax^{-1}}{\frac{1}{2} a^{1/2} x^{-3/2}}$

$\qquad = -2a^{-1/2} x^{3/2}.$

9·21. Chord of Curvature through the origin. Let BP be a curve and let PDL be the circle of curvature at P.



Let PT be the tangent at P. Then the angle between the radius vector OP and the tangent PT is φ.

Now the angle in a semi-circle is a right angle.

Hence, as is evident from the figure, PE, *the chord of curvature through the origin, is equal to*

$$2\rho \sin \varphi.$$

Ex. Find the chord of curvature through the pole of the curve $r^n = a^n \cos n\theta$. [*Allahabad*, 1927]

Differentiating logarithmically,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = -n \cdot \frac{\sin n\theta}{\cos n\theta}.$$

Therefore $\tan \phi = r\dfrac{d\theta}{dr} = -\cot n\theta.$

Hence $\sin \phi = -\cos n\theta.$

Also $\rho = a^n r^{-n+1}/(n+1)$. See § 9·15, example.

Hence the chord of curvature through the pole $= 2\rho \sin \phi$

$$= 2a^n r^{-n+1} (\cos n\theta)/(n+1)$$

$$= 2r/(n+1)$$

Examples

1. Find the centre of curvature of the following curves at the points indicated:

(i) $y = 3x^3 + 2x^2 - 3$ at $(0, -3)$,

(ii) $y = x^3 - 6x^2 + 3x + 1$ at $(1, -1)$

4 2. In the parabola $x^2 = 4ay$, prove that the coordinates of the centre of curvature are $(-x^3/4a^2,\ 2a + 3x^2/4a)$

3. Show that the centre of curvature at the point determined by t on the ellipse $x = a \cos t, y = b \sin t$, is given by

$$x = \frac{a^2 - b^2}{a} \cos^3 t,\ y = -\frac{a^2 - b^2}{b} \sin^3 t.$$

4. For the curve $a^2 y = x^3$, show that the centre of curvature (α, β) is given by

$$\alpha = \frac{x}{2}\left(1 - \frac{9x^4}{a^4}\right),\quad \beta = \frac{5}{2}\frac{x^3}{a^2} + \frac{a^2}{6x}.$$

5. In the curve $y = c \cosh (x/c)$, show that the coordinates of the centre of curvature are

$$X = x - y\{(y^2/c^2) - 1\}^{1/2},\ Y = 2y.$$

6. Show that the chord of curvature, through the pole, of the equiangular spiral $r = a\, e^{m\theta}$ is $2r$. [*Allahabad*, 1933]

7. Show that the chord of curvature through the pole for the curve $p = f(r)$ is $2f(r)/f'(r)$. [*Lucknow*, 1932]

8. Find the chord of curvature through the pole of the cardioid $r = a(1 + \cos\theta)$.

9. Show that the chord of curvature through the focus of a parabola is four times the focal distance of the point, and the chord of curvature parallel to the axis has the same length.

10. In the curve $y = a \log \sec (x/a)$, prove that the chord of curvature parallel to the axis of y is of constant length. [*Agra*, 1933]

9·3. Concavity and Convexity. Let P be a given

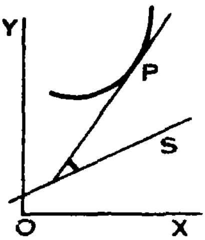


Fig. 1

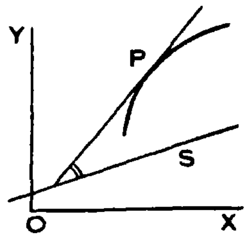
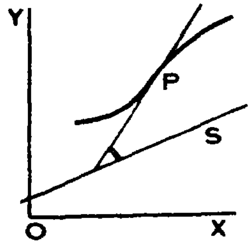


Fig. 2

point on a curve and S a given straight line which does not pass through P. Then the curve is said to be *concave* or *convex* at P with respect to S, according as a sufficiently small arc containing P lies entirely *within* or *without* the acute angle formed by S and the tangent to the curve at P.

Thus in Fig. 1 the curve at P is convex to S, and in Fig. 2 it is concave.

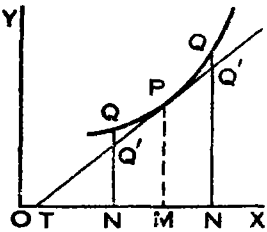
If the curve on one side of P is concave, and on the other side of P convex to any line S, the curve evidently crosses its tangent at P. Such a point is called a *point of inflexion*.

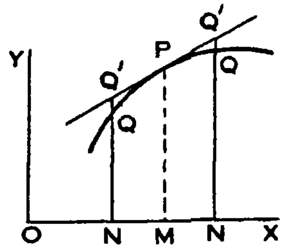


9·31. A test of concavity or convexity. We shall consider concavity and convexity with respect only to the axis of x.

Let the equation to the curve be $y = f(x)$ and let P be the point (x, y) and Q the point $(x + h, y + k)$.

Let the ordinate QN of Q meet the tangent to the curve at P in Q'.





Then, as the equation of the tangent is

$$Y - y = f'(x)(X - x),$$

it follows (by putting $X = x + h$) that

$$NQ' = y + f'(x)h.$$

Also, by Taylor's Theorem, if $0 < \theta < 1$,

$$NQ = f(x+h) = f(x) + hf'(x) + (1/2!)h^2 f''(x) + \dots$$
$$+ \{1/(n-1)!\}h^{n-1}f^{(n-1)}(x) + (1/n!)h^n f^{(n)}(x + \theta h).$$

Hence, by subtraction,

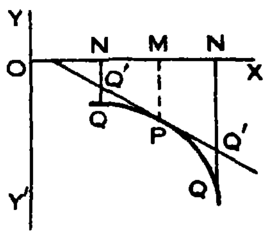
$$NQ - NQ' = h^2\{(1/2!)f''(x) + (1/3!)hf'''(x) + \dots$$
$$+ (1/n!)h^{n-2}f^{(n)}(x + \theta h)\}.$$

Now, if $f''(x)$ is not zero and we take h small enough, the sign of the right hand side will be the same as that of $f''(x)$ whether h is positive or negative.

Hence the curve is convex to the axis of x if $f''(x)$ is positive, and concave if $f''(x)$ is negative.

We have drawn the above figures for the case when the curve is above the axis of x. If, however, it is below, NQ and NQ' are both negative, and

$$NQ - NQ' = -\{|NQ| - |NQ'|\}.$$



Hence in this case the curve is convex to the axis of x if $|NQ| - |NQ'|$ be positive; i.e., if $NQ - NQ'$ be negative, i.e., if $f''(x)$ be negative.

Similarly the curve would be concave to the axis of x if $f''(x)$ be positive.

Both the cases are included in the statement :

A curve is convex or concave at P to the axis of x according as

$$y\frac{d^2y}{dx^2}$$

is positive or negative at P.

In case the higher differential coefficients of $f(x)$ do not exist, we can stop the Taylor's expansion at the term which involves h^2. It is only necessary that $f''(x)$ should exist and be continuous at the point under consideration. We have

$$NQ = f(x+h) - f(x) = hf'(x) + (1/2!)\,h^2 f''(x+\theta h).$$

Hence, $NQ - NQ' = (1/2!)\,h^2 f''(x+\theta h)$.

Now, if $f''(x)$ is not zero, and we take h small enough, the sign of the right hand side will, on account of the continuity at x of the second differential coefficient of $f(x)$, be the same as that of $f''(x)$. This will be so whether h is positive or negative.

Hence the preceding result will be true even if the third and the higher differential coefficients of $f(x)$ do not exist.

9·32. Test for Point of Inflexion. If, at P, $f''(x)$ is zero, but not $f'''(x)$, the above investigation shows that

$$NQ - NQ' = \frac{h^3}{3!}f'''(x) + \frac{h^4}{4!}f^{iv}(x) + \dots + \frac{h^n}{n!}f^{(n)}(x+\theta h).$$

For sufficiently small values of x the sign of the right-hand side is the same as that of $h^3 f'''(x)$, which changes sign when h changes sign. So the curve is concave to the axis of x on one side of P, and convex on the other. Hence, *there is a point of inflexion at* P, *if*

$$\frac{d^2y}{dx^2} = 0,$$

and $$d^3y/dx^3 \neq 0.$$

9·33. Points of undulation. If $f''(x) = f'''(x) = f^{iv}(x) = \dots f^{(n-1)}(x) = 0$, and $f^{(n)}(x) \neq 0$, it is easy to see from the value $NQ - NQ'$ that there would be a point of inflexion if n is odd.

If, however, n is even, the curve does not cross the tangent. Such a point (if n is greater than 2) is called a *point of undulation.*

To the eye a point of undulation appears just like an ordinary point.

9·34. Concavity and Convexity with respect to the axis of y. By considering y as the independent variable, we can easily show that a curve is convex or concave to the axis of y, according as $x(d^2x/dy^2)$ is positive or negative, and that, if d^2x/dy^2 is zero at P, but not d^3x/dy^3, there is a point of inflexion at P.

This test is useful when the tangent at P is parallel to the axis of y.

9·35. Concavity and Convexity with respect to a point.

Let P be a given point on a curve and let PT be the tangent at P to the curve. Then the curve is said to be concave or convex to a given point A, according as the curve in the immediate neighbourhood of P does, or does not, lie entirely on the same side of the tangent PT as A.

The investigation of § 9·31 gives us at once the following proposition:

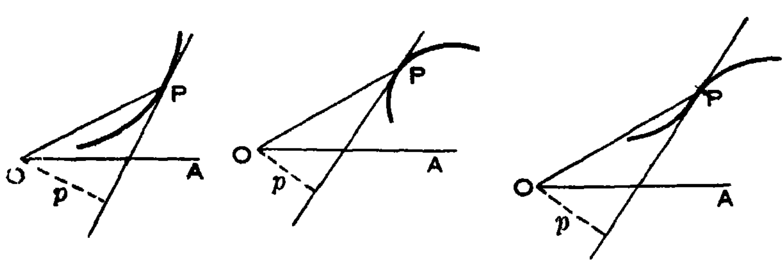
A given curve is convex or concave at P to the foot of the ordinate of P, according as yy'' is positive or negative.

9·36. Concavity and Convexity for Polar curves.

It is evident from the figures that if p, the perpendicular from the pole on the tangent, increases as r increases, then the curve is concave to the pole, i.e., *a curve is concave to the pole if dp/dr is positive.*

Similarly, *a curve is convex to the pole if dp/dr is negative.*

If dp/dr is zero at P, positive for points on one side of P and negative for the points on the other side of P, there must be a point of inflexion at P.



Curve concave to O. Curve convex to O. Point of inflexion at P.

But $\quad r\dfrac{dr}{dp}$ = radius of curvature $= \dfrac{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}$.

Hence $$\frac{dp}{dr} = \frac{r\left\{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}\right\}}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}.$$

It follows that *if*

$$\mathbf{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2r}{d\theta^2} = 0}$$

at P, there is, in general, a point of inflexion at P.

EXAMPLES

1. For the curve $y(a^2 - x^2) = x^3$, show that there is a point of inflexion at the origin, and also at the points for which $x = \pm a\sqrt{3}$.

2. Investigate the points of inflexion of the curve

$$y = (x - 2)^6 (x - 3)^5. \qquad \text{[Mysore, 1937]}$$

3. Show that the points of inflexion of the curve

$$y^2 = (x - a)^2 (x - b)$$

lie on the line $3x + a = 4b$. [*Lucknow*, 1934]

4. Show that every point in which the sine curve

$$y = c \sin (x/a)$$

meets the axis of x is a point of inflexion.

5. Is there a point of inflexion at $x = 0$ on the curve $y = x^4$?

6. Show that the curve $y = e^x$ is at every point convex to the foot of the corresponding ordinate.

7. In the curve $a^{m-1} y = x^m$, prove that the origin is a point of inflexion if m be odd and greater than 2.

8. For a curve given by its polar equation show that the points of inflexion are given by

$$u + \frac{d^2u}{d\theta^2} = 0,$$

where $u = 1/r$.

9. For the curve $(\theta^2 - 1)r = a\theta^2$, show that there is a point of inflexion at the point where $r = 3a/2$.

10. Show that the points of inflexion on the curve $r = b\theta^n$ are given by

$$r = b\{-n(n+1)\}^{n/2}.$$

9·4. Contact of curves. Let $y = \phi(x)$ and $y = \psi(x)$ be the equations to two curves.

Then, if $\phi(a) = \psi(a)$, the curves intersect at the point for which $x = a$. Let P be this point.

If $\phi(a) = \psi(a)$, and also $\phi'(a) = \psi'(a)$, the tangents to the two curves at P are the same; hence the two curves touch each other at P. We say that the curves have a *contact of the first order* at P, provided $\phi''(a) \neq \psi''(a)$.

If $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, and also $\phi''(a) = \psi''(a)$, the curves are said to have a *contact of the second order* at P, provided $\phi'''(a) \neq \psi'''(a)$.

In general, if $\phi(a) = \psi(a)$, $\phi'(a) = \psi'(a)$, ... $\phi^n(a) = \psi^n(a)$, and $\phi^{(n+1)}(a) \neq \psi^{(n+1)}(a)$, the curves are said to have a *contact of the nth order* at the point $x = a$.

9·41. The circle which has a contact of the second order. The equation of any circle is

$$(X-a)^2 + (Y-b)^2 = c^2. \qquad \ldots \quad (1)$$

If this has a contact of the second order with the curve $y = f(x)$ at the point (x, y), we must have

$$\left.\begin{aligned} X &= x \\ Y &= y \\ \frac{dY}{dX} &= \frac{dy}{dx} \\ \frac{d^2Y}{dX^2} &= \frac{d^2y}{dx^2} \end{aligned}\right\} \qquad \ldots \quad (2)$$

Now, by differentiating (1), we have

$$\left.\begin{aligned} (X-a) + (Y-b)\frac{dY}{dX} &= 0, \\ \text{and} \quad 1 + \left(\frac{dY}{dX}\right)^2 + (Y-b)\frac{d^2Y}{dX^2} &= 0. \end{aligned}\right\} \qquad \ldots \quad (3)$$

The equations (1) and (3) become, on substituting in them from (2),

$$(x-a)^2 + (y-b)^2 = c^2, \qquad \ldots \quad (4)$$

$$(x-a) + (y-b)\frac{dy}{dx} = 0, \qquad \ldots \quad (5)$$

and

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-b)\frac{d^2y}{dx^2} = 0. \qquad \ldots \quad (6)$$

These are the equations which determine a, b and c. The last of these equations gives

$$b - y = \frac{1 + \left(\dfrac{dy}{dx}\right)^2}{\dfrac{d^2y}{dx^2}}.$$

Then we have from (5), $a = x - \dfrac{\dfrac{dy}{dx}\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}{\dfrac{d^2y}{dx^2}}$.

These equations show that if the circle (1) has a contact of the second order with the curve $y = f(x)$ at (x, y), its centre must coincide with the centre of curvature of the curve at (x, y). Also equation (4) shows that the point (x, y) lies on the circle (1).

Hence *the circle which has a contact of the second order with the curve at (x, y) is the same as the circle of curvature of the curve at (x, y).*

9·42. Osculating Circle. *The circle which has a contact of the second order with the curve at (x, y) is also called the* **osculating circle** of the curve at (x, y). We have proved above that it is the same as the circle of curvature.

9·43. The crossing of curves. In § 9·3 we considered the case in which the curve crosses its tangent. We can now give a more general proposition as follows:

Two curves which have a contact of the nth order at the point for which $x = a$, cross each other, or do not cross each other, at that point according as n is even or odd.

Let the curves be $y = \phi(x)$ and $y = \psi(x)$.

The ordinates at $x = a + h$ for the two curves are $\phi(a + h)$ and $\psi(a + h)$ respectively. But, if $0 < \theta < 1$,

$$\phi(a + h) = \phi(a) + h\phi'(a) + \ldots + (1/n!)\,h^n \phi^{(n)}(a) + \ldots + \{1/(m-1)!\}\,h^{m-1}\phi^{(m-1)}(a) + (1/m!)\,h^m \phi^{(m)}(a + \theta h),$$

and
$$\psi(a + h) = \psi(a) + h\psi'(a) + \ldots + (1/n!)\,h^n \psi^{(n)}(a) + \ldots + \{1/(m-1)!\}\,h^{m-1}\psi^{(m-1)}(a) + (1/m!)\,h^m \psi^{(m)}(a + \theta h).$$

Subtracting, and remembering that the curves have a contact of order n at $x = a$, we have

$$\phi(a + h) - \psi(a + h) = \{1/(n+1)!\}\,h^{n+1}\{\phi^{(n+1)}(a) - \psi^{(n+1)}(a)\} + \{1/(n+2)!\}\,h^{n+2}\{\phi^{(n+2)}(a) - \psi^{(n+2)}(a)\} + \ldots$$

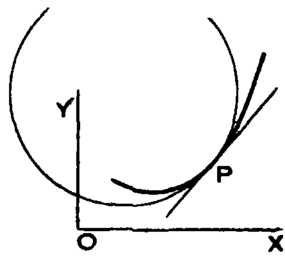
Now the sign of the right hand side, and therefore also of the left side, for sufficiently small values of h, is the same as that of

$$h^{n+1}\{\phi^{n+1}(a) - \psi^{n+1}(a)\}/(n+1)!\,;$$

and changes sign, or does not change sign, with h, according as n is even or odd. Hence the proposition.

NOTE. Of course m in the above must be supposed to be greater than n, but it may be taken to be equal to $n + 1$ if necessary on account of the non-existence of the higher differential coefficients But $\phi^{(n+1)}(a)$ and $\psi^{(n+1)}(a)$ must be continuous at $x = a$. See § 9·31.

9·44. The circle of curvature crosses the curve. We have seen that the circle which has a contact of the second order with the curve is the same as the circle of curvature Hence, in general, the circle of curvature at any point on the curve has a contact of the second order there, and will therefore cross the curve at that point



In exceptional cases the circle of curvature might have a contact of a higher order and so might not cross the curve

9·45. The circle of curvature crosses the curve. Alternative Proof.

The proposition of the last article can also be inferred from general reasoning. In general, the curvature will either go on increasing, or will go on decreasing, as, travelling along the curve, we pass through a given point P on it Hence the curvature at points on one side of P will be less, and on the other side greater, than the curvature at P. Thus on one side the curve will be outside, and on the other inside, the circle (§§ 9·1, 9·11). Hence, *in general, the circle of curvature at any point on the curve crosses the curve at that point.*

EXAMPLES

1. Show that the circle $(x - \frac{3}{4}a)^2 + (y - \frac{3}{4}a)^2 = \frac{1}{2}a^2$ and the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ have a contact of the third order at the point $(\frac{1}{4}a, \frac{1}{4}a)$.

2. Show that the curves

$$y = a_1x^n + b_1x^{n+1} + c_1x^{n+2}$$
$$\text{and } y = a_2x^n + b_2x^{n+1} + c_2x^{n+2}$$

have a contact of the $(n - 1)$th order at the origin.

Examples on Chapter IX

1. Prove that for the curve

$$s = a \log \cot (\tfrac{1}{4}\pi - \tfrac{1}{2}\psi) + a \sin \psi \sec^2 \psi,$$

$$\rho = 2a \sec^3 \psi,$$

and hence that
$$\frac{d^2y}{dx^2} = \frac{1}{2a},$$
and that this differential equation is satisfied by the parabola

$$x^2 = 4ay$$

2. For the rectangular hyperbola $xy = k^2$, prove that

$$\rho = \tfrac{1}{2}r^3/k^2,$$

r being the central radius vector of the point considered

[*Madras*, 1936]

3. In the curve $y = ae^{x/a}$, prove that

$$\rho = a \sec^2 \theta \operatorname{cosec} \theta, \text{ where } \theta = \tan^{-1}(y/a)$$

4. If ρ, ρ' be the radii of curvature at the extremities of two conjugate diameters of an ellipse, prove that

$$(\rho^{2/3} + \rho'^{2/3})\, a^{2/3} b^{2/3} = a^2 + b^2 \qquad \text{[*Allahabad*, 1926]}$$

5. Find the curvature at the point $(0, a)$ of the curve

$$(x^2 + y^2)^2 = a^2(y^2 - x^2) \qquad \text{[*Math Tripos*, 1934]}$$

6. If the coordinates of a point on a curve be given by the equations

$$x = c \sin 2\theta\, (1 + \cos 2\theta), \quad y = c \cos 2\theta\, (1 - \cos 2\theta),$$

show that the radius of curvature at the point is $4c \cos 3\theta$.

7. Show that in the curve

$$x = 3a\, (\sinh u \cosh u + u)/2, \quad y = a \cosh^3 u,$$

if the normal at $P\ (x, y)$ meets the axis of x in G, the radius of curvature at P is equal to $3\, PG$ [*Allahabad*, 1932]

8. The coordinates of a point on a curve are given by

$$x = a \sin t - b \sin (at/b), \quad y = a \cos t - b \cos (at/b)$$

Show that the equation of the tangent at the point whose parameter is t is

$$x \cos \frac{a+b}{2b} t - y \sin \frac{a+b}{2b} t + (a - b) \sin \frac{a-b}{2b} t = 0,$$

and that the radius of curvature at this point is

$$\frac{4ab}{a+b} \sin \frac{a-b}{2b} t \qquad \text{[*London*, 1934]}$$

9. Show that the curvatures of the curves $r = a\theta$ and $r\theta = a$ at their common point are in the ratio $3 : 1$

10. Prove that the distance between the pole and the centre of curvature corresponding to any point on the curve $r^n = a^n \cos n\theta$ is

$$\frac{\{a^{2n} + (n^2 - 1)\, r^{2n}\}^{1/2}}{(n + 1)\, r^{n-1}}. \qquad \text{[Math. Tripos, 1907]}$$

11. Show that in the rectangular hyperbola $r^2 \cos 2\theta = a^2$,

$$\varrho = r^3/a^2. \qquad \text{[Patna, 1937]}$$

12. Show that for the cardioid $r = a\,(1 - \cos\theta)$,

$$\varrho = (4a/3) \cos \tfrac{1}{2}\theta \propto \sqrt{r}. \qquad \text{[Allahabad, 1927]}$$

13. Show that for the curve in which $s = ae^{x/c}$,

$$c\,\varrho = s\,(s^2 - c^2)^{1/2}.$$

14. Prove that

$$\frac{1}{\varrho} = \frac{d}{dx}\left(\frac{dy}{ds}\right),$$

where ϱ is the radius of curvature and s is the length of the arc of the curve measured from some fixed point on it.

15. Show that the curve for which $y = ae^{m\frac{x}{a}}$, the radius of curvature is m times the tangent.

16. Find the radii of curvature at the origin in the curve

$$a(y^2 - x^2) = x^3.$$

[Transforming the equation to the internal and external bisectors of the angle between the axes, it becomes

$$4axy\sqrt{2} = (x - y)^3;$$

hence the radii of curvature are $2a\sqrt{2}$, and $-2a\sqrt{2}$, respectively.—*Williamson*]

17. Prove that for any curve

$$\frac{r}{\varrho} = \sin\phi\left(1 + \frac{d\phi}{d\theta}\right),$$

where ϱ is the radius of curvature and $\tan\phi = r\,d\theta/dr$.

[*Allahabad*, 1937]

18. Prove that, for a curve defined by its (p, ψ) equation, the projection of the radius vector on the tangent is $dp/d\psi$, and that the radius of curvature is $p + d^2p/d\psi^2$. Explain the convention of sign in each case.

The line which joins the origin to Q, the centre of curvature at a point P on the curve $p = a \sin n\psi$, meets the tangent at P in T. Prove that $OQ = n^2\, OT$. [*Math. Tripos*, 1923]

19. The curve $r = a\, e^{\theta \cot \alpha}$ cuts any radius vector in the consecutive points $P_1, P_2, \ldots P_n, P_{n+1}, \ldots$. If ϱ_n denotes the radius of curvature at P_n, prove that

$$\frac{1}{m - n} \log \frac{\varrho_m}{\varrho_n}$$

is constant for all integral values of m and n. [*Math. Tripos*, 1930]

20. If α, β be the coordinates of the centre of curvature of the curve $y^3 = a^2x$, show that

$$\alpha = (a^4 + 15y^4)/6ya^2, \beta = (a^4y - 9y^5)/2a^4.$$

21. Find the coordinates of the centre of curvature of the catenary $y = a \cosh (x/a)$, and show that the radius of curvature is equal in length to the portion of the normal intercepted between the curve and the axis of x.

22. Prove that the curvature at a point of the curve $y = f(x)$ is given by $(d^2y/dx^2) \cos^3 \psi$, where ψ is the inclination of the tangent at the point to the axis of x.

Prove that the coordinates of the centre of curvature at any point (x, y) can be expressed in the form

$$x - \frac{dy}{d\psi} \text{ and } y - \frac{dx}{d\psi}.$$ [*Nagpur*, 1930]

23. Prove that the common chord of the parabola and the circle of curvature at any point is of length $8\sqrt{\{r(r-a)\}}$, where r is the distance of the point from the focus of the parabola.

24. Show that the length of the chord of curvature, parallel to the axis of y, at the origin, in the parabola

$$y = mx + x^2/a,$$

is $(1 + m^2)a$, and the equation of the circle of curvature is

$$x^2 + y^2 = (1 + m^2)\, a\, (y - mx).$$

25. Show that in the catenary $y = a \cosh (x/a)$ the chord of curvature parallel to the axis of y is double of the ordinate, and that parallel to the axis of x is of length $a \sinh (2x/a)$.

26. Show that in any curve the chord of curvature perpendicular to the radius vector is

$$2\rho (r^2 - p^2)^{1/2}/r.$$

27. Show that the curve $x^3 + y^3 = a^3$ has inflexions where it cuts the coordinate axes.

28. In the curve $y\,(a^2 + x^2) = 2x^2$, prove that for varying values of a the locus of the points of inflexion is the straight line $y = \frac{1}{2}$.

29. Prove that the centres of curvature at points of a cycloid lie on an equal cycloid. The coordinates, referred to suitable axes, of a point on a cycloid may be assumed to be

$$x = a\,(t + \sin t),\ y = a\,(1 + \cos t).$$

[*Math. Tripos*, 1935]

30. Show that the circle of curvature at the point $(am^2, 2am)$ of the parabola $y^2 = 4ax$ has for its equation

$$x^2 + y^2 - 6am^2 x - 4ax + 4am^3 y - 3a^2m^4 = 0,$$

and determine the point where it meets the curve again.

[*Allahabad*, 1925]

CHAPTER X

SINGULAR POINTS. CURVE TRACING

10·1. Multiple-valued Functions. If y has two or more values for every value of x, it is usually possible to suppose that this is a case where two or more distinct functions are given (cf. § 1·12). But it is generally more convenient to regard the curves corresponding to these distinct functions, not as different curves, but as different *branches* of one curve. It is customary, moreover, to call y in such cases a *multiple-valued function* of x.

Thus if $y = x \pm \sqrt{(x^2 + 1)}$, y will be called a double-valued function of x. Also, $x + \sqrt{(x^2 + 1)}$ and $x - \sqrt{(x^2 - 1)}$ are called the two branches of the function y.

10·11. Multiple Points. A point of inflexion is a *singular point* (i.e., an unusual point) on the curve, for the tangent does not usually cross the curve, as it does at a point of inflexion. But there are other kinds of singularities. A point, for example, may be a multiple point.

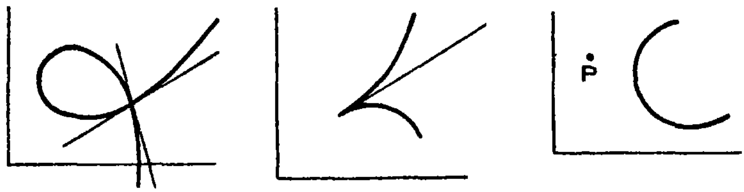
A point through which more than one branches of a curve pass is called a *multiple point* on the curve.

A point on a curve is called a *double point* if two branches of the curve pass through it; a *triple point* if three branches pass through it and so on. If r branches pass through the point, it is called a multiple point of the rth order.

If the two branches through a double point on a curve are real, and the tangents to them are not coincident, the double point is called a *node*.

If the two tangents are coincident, the point is called a *cusp*.

If there are no real points on the curve in the neighbourhood of a point P on the curve, P is called a *conjugate point* (or an *isolated point*). The process of finding the



Node. Cusp. Conjugate Point.

tangent usually gives imaginary tangents at such a point, but sometimes the tangents come out real.

Conjugate points really arise from the fact that an imaginary expression $A + iB$ becomes real when $A = 0$, and $B = 0$. Thus the imaginary "straight line"

$$x - 2 + i(y - 4) = 0 \quad . \quad . \quad . \quad . \quad (1)$$

has one real point on it, viz., the point $(2, 4)$.

This is a conjugate point on the straight line.

The student must carefully notice that *any and every imaginary point does* **not** *lie on the imaginary straight line under consideration*. It is only those points whose coordinates satisfy (1) that lie on it. Thus, as substitution will immediately show, $(3 - i, 5 + i)$ lies on (1), but not $(4 + i, 3 + i)$. A similar statement is true for every curve having an imaginary branch. That is why the analytical process of finding the tangent at a conjugate point gives us a definite tangent; only the constants occurring in the tangent involve $\sqrt{(-1)}$.

The student must not interpret "imaginary" as something which does not exist. The tangent to the curve of $y = \sin(1/x)$ at $x = 0$ does not exist. But it would be wrong to say that the tangent is imaginary there.

If we were to discard the use of negative numbers, $3 - 4$ will have no meaning; i.e., we shall not be able to find a number which will be equal to $3 - 4$. If, however, we introduce negative numbers $3 - 4$ has a perfectly definite meaning. Similarly, if we have only the real numbers, $3 + 4\sqrt{(-1)}$ will have no meaning. But if we introduce the complex numbers, expressions like $3 + 4\sqrt{(-1)}$ will have a perfectly definite meaning. Such expressions would no more be imaginary in the popular sense than $3 - 4$*. In fact the word

*The great mathematician Descartes (after whom the Cartesian coordinates have been named) used to call negative roots of equations *false* roots.

"imaginary" is used in mathematics merely in the sense of "involving the square root of -1" and not in the sense of something which does not exist.

Ex. Find the nature of the origin on the curve

$$a^4y^2 = x^4(x^2 - a^2).$$

The origin lies on the curve; but

$$y = \pm\frac{x^2}{a^2}\sqrt{(x^2 - a^2)},$$

so that the values of y are imaginary, whether x is positive or negative, when x is small but $\neq 0$. Hence the origin is a conjugate point on the curve.

Also $$\frac{dy}{dx} = \pm\left\{\frac{2x}{a^2}\sqrt{(x^2 - a^2)} + \frac{x^2}{a^2}\frac{x}{\sqrt{(x^2 - a^2)}}\right\}.$$

Hence $\dfrac{dy}{dx} = 0$ at $(0,0)$. So by § 7·12, the tangent at $(0,0)$ is

$$Y - 0 = 0\,(X - 0),$$

i.e., $$Y = 0,$$

which is *real*, showing that the tangent may be real at a conjugate point.

[If we apply § 10·2, then also we get for the tangents at the origin the equation $Y^2 = 0$, i.e., $Y = 0$.]

10·12. Species of Cusps. A cusp might be single or double, according as the curve lies entirely on one side of the normal or on both sides. Also, a single cusp might be of the *first species* or the *second*, according as both the branches lie on *opposite* sides or on the *same* side of the tangent. Hence we have the following types of cusps:

Single cusp of the first species (Fig. 1).

Fig. 1

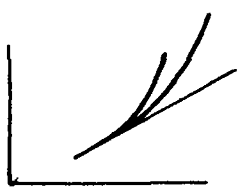


Fig. 2

Single cusp of the second species (Fig. 2).

Double cusp of the first species (Fig. 3).

Double cusp of the second species (Fig. 4).

Double cusp with change of species, or a point of *osculinflexion* (Fig. 5).

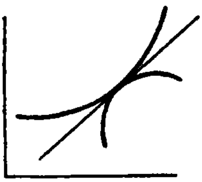


Fig. 3

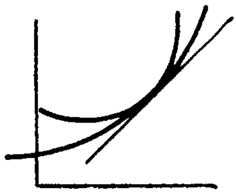


Fig. 4

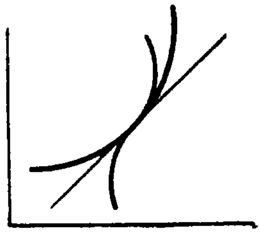


Fig. 5

A cusp of the first species is also called a *keratoid* cusp (i.e., a cusp like *horns*), and a cusp of the second species a *ramphoid* cusp (i.e., a cusp like a *beak*).

10·2. Tangents at the origin. In order to investigate the nature of a multiple point it is necessary to find the tangent or tangents there. The following proposition is very helpful in this :

If a curve passes through the origin and is given by a rational, integral, algebraic equation, the equation to the tangent or tangents at the origin is obtained by equating to zero the terms of the lowest degree in the equation to the curve.

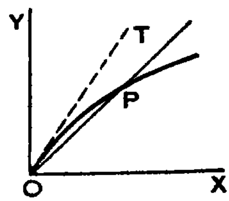
Let the equation to the curve be

$$a_1x + a_2y + b_1x^2 + b_2xy + b_3y^2 + c_1x^3 + \ldots = 0, \quad (1)$$

the constant term being absent, as the curve, by hypothesis, passes through the origin.

Let P be the point (x, y). Then the equation to OP is

$$Y = (y/x)X.$$



The equation of the tangent at O is, therefore (§ 7·1),

$$Y = \{\lim_{x\to 0}(y/x)\}X. \quad . \quad . \quad (2)$$

Let us exclude for the present the case when the tangent is the y-axis (viz., when $\lim_{x\to 0}(y/x) = \pm\infty$).

I. Let $a_2 \neq 0$. Dividing by x and taking limits as $x \to 0$, we get

$$a_1 + a_2\{\lim_{x\to 0} (y/x)\} = 0. \qquad . \quad . \quad . \quad (3)$$

Substituting the value of $\lim_{x\to 0} (y/x)$ from this in (2), or what comes to the same thing, eliminating $\lim_{x\to 0} (y/x)$ between (2) and (3), we have

$$a_1X + a_2Y = 0;$$

or, since there is now no danger of confusion, writing x, y for, X, Y, the tangent at the origin to (1) is

$$a_1x + a_2y = 0,$$

which could have been obtained by equating to zero, the terms of the lowest degree in (1).

If $a_2 = 0$, then by (3), $a_1 = 0$, and we get the next case.

II. Let $a_1 = 0$ and $a_2 = 0$, but suppose that b_2 and b_3 are not both zero.

Divide (1) by x^2, and take limits. If $\lim_{x\to 0} (y/x)$ be denoted by m, we see that

$$b_1 + b_2 m + b_3m^2 = 0, \qquad . \quad . \quad . \quad . \quad (4)$$

which shows that there are in general two values of m, and therefore two tangents at the origin. Their equation is obtained by eliminating m between (4) and (2), and is

$$b_1x^2 + b_2xy + b_3y^2 = 0.$$

This equation could have been written down by equating to zero the terms of the lowest degree in the equation to the curve (viz. (1) with $a_1 = 0$, $a_2 = 0$).

If $b_2 = b_3 = 0$, then by (4), $b_1 = 0$.

III. If $a_1 = a_2 = b_1 = b_2 = b_3 = 0$, we can show similarly that the rule still holds; and so on.

If the tangent at the origin is the y-axis, we can easily see, by supposing the axes of x and y to be interchanged for an instant, that the rule is still true.

Hence by equating to zero the terms of the lowest degree, all the tangents will be obtained, including the y-axis if it is a tangent.

10·21. Nature of a cusp at the origin when the axis of x is a tangent. If the common tangent at the origin to the two branches of the curve is the axis of x, solve for y.

The reality or otherwise of the roots for positive and negative values of x, also the fact of the roots being of the same or of opposite signs, will decide what type of cusp there is at the origin.

Since we want only the approximate shape of the curve in the immediate neighbourhood of the origin, the terms which are very small for small values of x, viz., terms of high orders in x, may be neglected in deciding the nature of the cusp, provided that this does not reduce the equation merely to that of the tangent, or does not make the two branches of the curve coincide. For the same reason moreover, and because there are only two branches of the curve through the origin (as we know from the equation to the tangents at the origin), terms involving powers of y above the second can also be neglected.

Ex Examine the nature of the origin on the curve

$$a^3y^2 - 2abx^2y + x^5 + cx^6 = 0$$

Here the term of the lowest degree equated to zero gives $y^2 = 0$, showing there are two coincident tangents ($y = 0$) at the origin.

Solving for y, after neglecting x^6, we have

$$y = \frac{abx^2 \pm \sqrt{(a^2b^2x^4 - a^3x^5)}}{a^3}. \quad \ldots \quad (1)$$

When x is very small numerically, $a^2b^2x^4 - a^3x^5$ has the same sign as $a^2b^2x^4$, which is positive whether x be positive or negative. Thus y is real for numerically small values of x (positive or negative). Hence we have a double cusp.

Also, when x is positive and small, $a^2b^2x^4 - a^3x^5 < a^2b^2x^4$,

i.e.,
$$+\sqrt{(a^2b^2x^4 - a^3x^5)} < abx^2$$

Hence, when x is positve and small, both the values of y are positive; i.e., on the right of the origin the curve is of the second species.

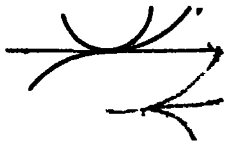
Again, when x is negative, and numerically small,

$$+\sqrt{(a^2b^2x^4 - a^3x^5)} > abx^2,$$

because now $-a^3x^5$ is positive.

Thus now one value of y as given by (1) is positive and the other negative. Hence there is a cusp of the first species on the left of the origin.

Thus there is an osculinflexion at the origin.



10·22. Nature of a cusp at the origin. Other cases. If the tangent at the origin is the y-axis, we can proceed as in the last article, solving for x instead of y.

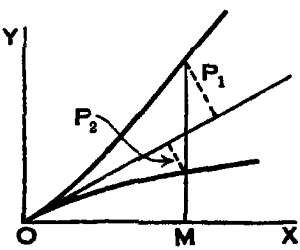
But if the tangent is neither the x-axis, nor the y-axis, but, say, the line

$$aX + bY = 0,$$

then the perpendicular P_1 from the point (x, y) on the curve is

$$\frac{ax + by}{\sqrt{(a^2 + b^2)}},$$

which is proportional to $ax + by$. Let us put $P = ax + by$. If we eliminate y between this and the equation to the curve, i.e., if we substitute $(P - ax)/b$ for y in the equation to the curve, we get a relation between P and x. Solving for P, we shall be able to decide, from the sign and the reality of the values of P for positive and negative value of x, the nature of the cusp. Thus, if its values are of opposite signs (as in the figure) for a positive small x, the perpendiculars from those points which have this value of x as abscissa, on the tangent $ax + by = 0$, are of opposite signs. Hence there is a cusp of the first species on the right; and so on.



Ex. Examine the nature of the origin on the curve

$$(2x + y)^2 - 6xy\,(2x + y) - 7x^3 = 0.$$

The tangents at the origin are $(2x + y)^2 = 0$.

Putting $P = 2x + y$ we have from the equation to the curve

$$(1 - 6x)P^2 + 12x^2P - 7x^3 = 0.$$

Therefore
$$P = \frac{-6x^2 \pm \sqrt{\{36x^4 + 7x^3(1 - 6x)\}}}{1 - 6x}$$
$$= \frac{-6x^2 \pm \sqrt{(7x^3 - 6x^4)}}{1 - 6x}.$$

When x is negative, the values of P are imaginary; and when x is positive, the values of P are real and of opposite signs (because, when x is small, $x^{3/2}$ is greater than x^2).

Hence there is a single cusp of the first species at the origin.

10·23. Cusp at any point. In order to find the nature of the cusp at any point, it is usual to transfer the origin to that point and then apply the methods of the preceding articles.

10·24. Search for double points. In order to search for double points on a curve, transfer the origin to the point (h, k). In the transformed equation the constant term and the terms of the first degree must be absent in order that the new origin may be a double point. This gives three equations, which are more than sufficient to determine h and k. Take any two of these and solve for h and k. If the values obtained satisfy the remaining equation, the point (h, k) will be a double point. If no set of values of h and k can be found to satisfy all the three equations, there is no double point anywhere on the curve.

Ex. Determine the existence and nature of the double points on the curve

$$(x-2)^2 = y(y-1)^2. \qquad \text{[\textit{Allahabad}, 1933]}$$

Transferring the origin to (h, k), the equation becomes

$$(x+h-2)^2 - (y+k)(y+k-1)^2 = 0,$$

or $$(h-2)^2 + 2x(h-2) - k(k-1)^2 - y\{2k(k-1) + (k-1)^2\}$$
$$+ \text{ terms of higher degree} = 0.$$

Hence, for a double point

$$(h-2)^2 - k(k-1)^2 = 0, \qquad \dots\dots (1)$$

$$h - 2 = 0, \qquad \dots\dots (2)$$

and $$2k(k-1) + (k-1)^2 = 0. \qquad \dots\dots (3)$$

Eqn. (2) gives $h = 2$, and (3) gives $k = 1$, or $\frac{1}{3}$.

By substitution in (1) we see that $h = 2$, $k = 1$, satisfies it, but not $h = 2$, $k = \frac{1}{3}$. Hence there is a double point only at $(2, 1)$.

Transferring the origin to it, the equation becomes

$$x^2 = (y+1)y^2, \text{ or } x = \pm y\sqrt{(y+1)}.$$

When y is positive and small, x has two real values, one positive and the other negative. When y is negative, but numerically small, the same is still true.

The tangents at the origin are $x^2 = y^2$.

Hence there is a node at the point $(2, 1)$ on the given curve.

10·25. A necessary condition for the existence of double points. If $f(x, y)$ is an expression involving x and y, and the equation to a curve is $f(x, y) = 0$, then on transferring the origin to (h, k), the equation will become

$$f(x+h,\ y+k) = 0.$$

It will be shown in the next chapter that, if $\frac{\partial f}{\partial x}$ means the differential coefficient of $f(x, y)$ when y is regarded as a constant, $\frac{\partial f}{\partial y}$ the differential coefficient when x is regarded as constant and $\frac{\partial^2 f}{\partial x^2}$ means $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$, and so on, then

$$f(x+h,\ y+k) = f(h, k) + x\left(\frac{\partial f}{\partial x}\right)_{x=h,\ y=k} + y\left(\frac{\partial f}{\partial y}\right)_{x=h,\ y=k}$$
$$+\tfrac{1}{2}\left\{x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=h,\ y=k} + 2xy\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{x=h,\ y=k} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=h,\ y=k}\right\}$$
$$+ \text{terms of higher degree.}$$

Hence, by the previous article, we see that at a double point we must have

$$\frac{\partial f}{\partial x} = 0,\quad \frac{\partial f}{\partial y} = 0,\quad f = 0.$$

If these conditions are satisfied, the tangents at the origin are given by

$$x^2\left(\frac{\partial^2 f}{\partial x^2}\right)_{x=h,\ y=k} + 2xy\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)_{x=h,\ y=k} + y^2\left(\frac{\partial^2 f}{\partial y^2}\right)_{x=h,\ y=k} = 0.$$

Hence, by § 10·11, *in general there will be a node, cusp, or conjugate point at the origin, according as there*

$$\left(\frac{\partial^2 f}{\partial x\,\partial y}\right)^2 >,\ =,\ \text{or} < \left(\frac{\partial^2 f}{\partial x^2}\right)\left(\frac{\partial^2 f}{\partial y^2}\right).$$

EXAMPLES

1. Write down the equation to the tangents at the origin for the following curves :

(i) $x^4 + 3x^3y + 2xy - y^2 = 0$,

(ii) $x^3 + 3xy + 7x^2 = 0$.

2. Show that the curve $x^4 - 2ax^2y - axy^2 + a^2y^2 = 0$ has a cusp of the second kind at the origin.

3. Show that the origin is a conjugate point on the curve

$$x^4 - ax^2y + axy^2 + a^2y^2 = 0. \qquad [\textit{Mysore, } 1936]$$

4. Show that the curve $y^3 = (x - a)^2 (2x - a)$ has a single cusp of the first kind at the point $(a, 0)$.

5. Show that the curve $(xy + 1)^2 + (x - 1)^3 (x - 2) = 0$ has a single cusp of the first kind at the point $(1, -1)$.

6. Show that the curve $y - b = (x - a)^{1/3} + (x - a)^{3/4}$ has a cusp of the second kind at the point $x = a$.

7. Find the coordinates of the double points on the curve

$$x^4 - 2ay^3 - 3a^2y^2 - 2a^2x^2 + a^4 = 0.$$

8. Briefly explain the meaning and use of d^2y/dx^2 in analysis.

Prove that, if $y^3 + 3ax^2 + x^3 = 0$, then

$$\frac{d^2y}{dx^2} + \frac{2a^2x^2}{y^5} = 0.$$

Show that the curve given by the above equation (i) is everywhere concave to the x-axis, (ii) has a point of inflexion at $x = -3a$, (iii) has a cusp at $x = 0$, and (iv) has an asymptote $x + y + a = 0$. [*Nagpur*, 1930]

9. Show that the curve $y^2 = bx \tan (x/a)$ has a node or a conjugate point at the origin according as a and b have like or unlike signs. $4 \equiv y$

[Hint : Expand $\tan (x/a)$ and retain only the first few terms].

10. Determine the position and character of the double points on

(i) $x^4 + y^3 + 2x^2 + 3y^2 = 0$,

(ii) $y (y - 6) = x^2 (x - 2)^3 - 9$, [*Agra*, 1935]

(iii) $x^3 - y^2 - 7x^2 + 4y + 15x - 13 = 0$. [*Agra*, 1936]

11. Show that the points of intersection of the curve

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

with the axes are cusps of the first species.

12. Examine the nature of the double points on the curve

$$(x + y)^3 - (\sqrt{2}) (y - x + 2)^2 = 0.$$

10·3. Curve Tracing. Cartesian Equations. The object of Curve Tracing is to find the approximate shape of a curve without the labour of plotting a large number of points.

If the Cartesian Equation is given, the student will find that he can invariably *solve it either for y, or for x, or for*

r (in terms of θ in the last case), otherwise the curve will be too difficult for him to trace.

Only curves in which we can solve for y need be considered here; because, if the equation cannot be solved for y, but can be solved for x, we have only to regard y as the independent variable. If the equation can be solved for r, the rules for tracing polar curves will apply.

10·31. Procedure.

1. Notice if the curve is symmetrical about any line, by applying the following rules, whose truth is evident:

(i) If the powers of y which occur in the equation are all even, the curve is symmetrical about the axis of x.

(ii) If the powers of x are all even, the curve is symmetrical about the axis of y. A curve might, of course, be symmetrical about both axes.

(iii) If x and y can be interchanged without altering the equation, the curve is symmetrical about the line $y = x$.

(iv) If on changing the signs of x and y both, the equation to the curve is not altered, the curve after being turned through two right angles will coincide with its old trace. (This is generally denoted by saying that there is symmetry in opposite quadrants).

2. Notice if the curve passes through the origin. If it does, write down the equation of the tangent, or tangents, there. If the origin is a singular point, find its nature.

3. Solve for y (which by supposition, is possible). Choose any convenient value of x for which y is finite, and if possible zero (generally $x = 0$ is convenient). Consider how y will vary as x increases and then tends to infinity, paying particular attention to those values of x for which $y = 0$, or $\to \infty$.

If the curve is symmetrical about the x-axis, or if there is symmetry in opposite quadrants, only positive values of y need be considered. The curve for negative values of y can be drawn from symmetry.

4. Starting from the chosen value of x, repeat the above procedure as x decreases and then tends to $-\infty$.

Of course, if the curve is symmetrical about the y-axis, it can be drawn for negative values of x by symmetry; so such values of x need not be considered afresh.

5. In the above procedure if y is found to be imaginary for a certain range of values of x, say for values of x betwen a and b, it would mean that the curve does not exist in the region bounded by the lines $x = a$, $x = b$.

6. If the procedure of paragraphs 3 and 4 shows that the curve extends to infinity, and there is approximately a linear relation between x and y for numerically large values of x, there is an oblique asymptote This should now be found, and also, if necessary, it should be investigated on which side of it the curve lies.

NOTE When x and y are numerically very large, only the highest powers of these may be retained to find the approximate shape of the curve

The presence of asymptotes parallel to the axes, and their position will become evident by the procedure of paras. 3 and 4.

7. Find the coordinates of a few points on the curve if it appears necessary.

For example, if y is 0 at $x = a$, and again at $x = b$, and is positive for the intermediate values, it might be desirable to find the maximum (greatest) value of y between a and b At the point for which y is maximum, the tangent (as is evident from geometry) will be horizontal and so dy/dx will be zero. Hence this point can be easily found (the subject is treated in detail in Chapter XIII) Even if the maximum value is not found, it would be desirable to find the value of y when x is equal to, say, $\frac{1}{2}(a + b)$.

8. If the curve as traced appears to possess a point of inflexion, that point can now be more accurately located by putting d^2y/dx^2 or d^2x/dy^2 equal to zero and solving the equation thus obtained.

9. *The student should remember that merely a knowledge of symmetry, asymptotes, tangents at the origin, points of inflexion, double points, and the coordinates of a few other points will never enable him to trace a curve. His difficulties regarding curve-tracing will vanish only if he realises that we have solved for y and expressed it as a function of x whose values can be easily found*

for every value of x. By noticing how y varies as x is made to vary continuously from $-\infty$ to ∞, the curve is easily traced.

We need not begin from $-\infty$ (if that is inconvenient) provided later we consider the remaining values also of x. Further we need not consider the actual value of y for every value of x: its values for a few values of x, coupled with the knowledge that for the intermediate values it is positive or negative, increasing or diminishing, would be enough, as the following examples will show:

NOTE. An equation of the second degree in x and y gives merely one of the conic sections, and so can be easily traced.

Ex. 1. Trace the curve

$$y^2(a+x) = x^2(3a-x).$$

(i) This curve is symmetrical about the axis of x.

(ii) The curve passes through the origin. The tangents there are given by $y^2 = 3x^2$, which represents two non-coincident straight lines. Hence we may expect a node at the origin.

(iii) Solving for y, and *considering only the positive value*,

$$y = x\sqrt{\left(\frac{3a-x}{a+x}\right)}. \qquad \ldots \ldots \quad (1)$$

If $x = 0$, then $y = 0$. When x is positive and small, y is real. We notice also that as

$$\frac{3a-x}{a+x} < \frac{3a}{a}, \text{ i.e., } < 3,$$

y is less than $x\sqrt{3}$. Hence the curve lies *below* the tangent $y = x\sqrt{3}$ for small positive values of x.

As x goes on increasing, y next becomes zero at $x = 3a$. When x is greater than $3a$, the expression under the radical sign is negative and so y is imaginary. Hence the curve in the first quadrant is probably of the shape shown in Fig. 1.

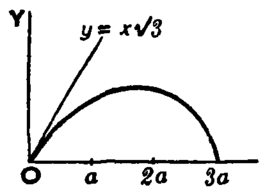


Fig. 1

To trace the curve more exactly we find the following also:

When $x = a$, $y = a$; and when $x = 2a$, $y = 2a/\sqrt{3} = 1{\cdot}2\,a$ nearly.

Also if we transfer the origin to $(3a, 0)$, the equation to the curve will become $y^2(4a+x) = (x+3a)^2(-x)$, and the tangent at

the new origin will be $x = 0$, obtained by equating to zero the terms of the lowest degree.

Hence the curve must be of the shape shown above.

(iv) if x is negative and numerically small, (1) shows that y is real. Also, for values of x under consideration, $3a - x > 3(a + x)$. Hence y is numerically greater than $-(\sqrt{3})x$, i.e., the curve lies above the tangent $y = -(\sqrt{3})x$ in the second quadrant.

As we move still more to the left, $a + x$ gets still smaller and so y gets larger. In fact as $x \to -a$ from the right of the point $x = -a$, the positive value of y tends to $+\infty$.

$x + a = 0$ is evidently an asymptote (§ 8·24).

When $x < -a$, the quantity under the radical is negative and so y is imaginary.

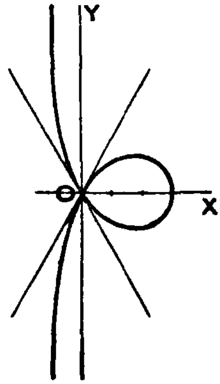


Fig. 2.

Therefore, taking symmetry into account, the complete curve must be as shown in figure 2.

Ex. 2. Trace the curve

$$y^2(x^2 + y^2) - 4x(x^2 + 2y^2) + 16x^2 = 0$$

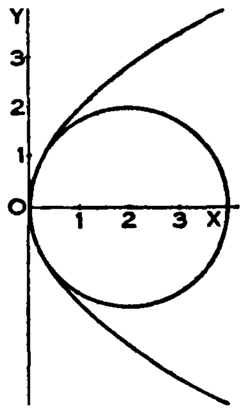
[*U. P., P. C. S.*, 1933]

(i) The curve is symmetrical about the x-axis.

(ii) The curve passes through the origin, the tangents there being $x^2 = 0$, which represents two coincident straight lines. Hence we may expect a cusp there.

(iii) The equation to the curve is a quadratic in y^2, and can be written as

$$y^4 + y^2(x^2 - 8x) - 4x^3 + 16x^2 = 0.$$



Hence $\quad y^2 = \frac{1}{2}\{8x - x^2 \pm \sqrt{(x^4 - 16x^3 + 64x^2 + 16x^3 - 64x^2)}\}$

$\quad = \frac{1}{2}\{8x - x^2 \pm x^2\}$

$\quad = 4x$, or $4x - x^2$.

Hence the curve consists of the parabola $y^2 = 4x$, and the circle $x^2 + y^2 = 4x$.

Ex. 3. Trace the curve

$$x = (y - 1)(y - 2)(y - 3). \qquad [\textit{Allahabad}, 1933]$$

(i) The curve is not symmetrical about the axes or about $x = y$.

(ii) It does not pass through the origin.

(iii) It is difficult to solve it for y. But it is already solved for x. Hence we take y as the independent variable.

When $y = 0$, $x = -6$.

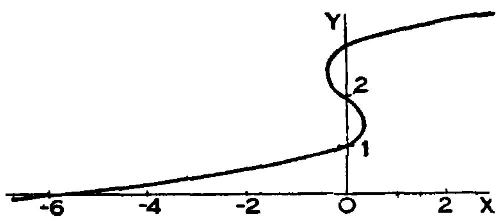
When $y = 1$, $x = 0$. Between $y = 0$ and $y = 1$, x is negative, as then all the three factors are negative.

When y lies between 1 and 2, x is positive as one factor is positive and two are negative. x next becomes zero at $y = 2$.

Between $y = 2$ and $y = 3$, x is negative.

x next becomes zero at $y = 3$.

When $y > 3$, x is positive. As $y \to \infty$, $x \to \infty$. For very large values of y, x is approximately equal to y^3. Hence there is no linear asymptote for this branch.



(iv) When y is negative, x is negative. As $y \to -\infty$, $x \to -\infty$. As in the last paragraph, we can see that there is no linear asymptote for this branch also.

(v) When $x = 1\frac{1}{2}$, $y = \frac{3}{8}$; when $x = 2\frac{1}{2}$, $y = -\frac{3}{8}$.

Hence the shape of the curve is as shown.

[If we like we can also find where the tangent is parallel to the y-axis. At these points $dx/dy = 0$, i.e.,

$$(y - 1)(y - 2) + (y - 1)(y - 3) + (y - 2)(y - 3) = 0,$$

or

$$3y^2 - 12y + 11 = 0,$$

i.e., the tangent is parallel to the y-axis where

$$y = \frac{6 \pm \sqrt{(36 - 33)}}{3} = 2 \pm \sqrt{3}/3 = 2{\cdot}6, \text{ and } 1{\cdot}4 \text{ nearly.}$$

We can now find the values of x for these values of y, and thus find the shape of the curve a little more exactly.]

Ex. 4. Trace the curve

$$y^2 = \frac{x(x-a)(x-2a)}{x+3a} \qquad [\textit{Allahavad}, 1927]$$

(i) The curve is symmetrical about the axis of x

(ii) The curve passes through the origin, the tangent there being $x = 0$.

(iii) The equation is already solved for y. Considering only positive values of y, we see that

When $x = 0, y = 0$

When $x > 0$, but small, y is real

When $x = a$, $y = 0$ again [When $x = \frac{1}{2}a$, $y = \frac{1}{3}a$ nearly]

Between $x = a$ and $x = 2a$, y^2 is negative and so y is imaginary.

When $x = 2a$, $y = 0$ When $x > 2a$, y is real The tangent at $(2a, 0)$ is (as we can see mentally by supposing the origin to be transferred to it) parallel to the y axis [When $x = 3a, y = a$, when $x = 4a$, $y = 1\cdot 8a$] As $x \to \infty$, $y \to \infty$ When x and y are large, we can write the equation as

$$\frac{y^2}{x^2} = \frac{\left(1 - \frac{a}{x}\right)\left(1 - \frac{2a}{x}\right)}{\left(1 + \frac{3a}{x}\right)}$$

$$= 1 - \frac{6a}{x} + \frac{20a^2}{x^2} + \ldots$$

Hence $$\frac{y}{x} = \pm\left(1 - \frac{3a}{x} + \frac{11a^2}{2x^2} + \ldots\right), \qquad \ldots (1)$$

or $$y = x - 3a + \frac{11}{2}\frac{a^2}{x} + \ldots,$$

considering only positive values of y/x.

Hence $y = x - 3a$ is an asymptote, and in the first quadrant the curve lies above the asymptote (§ 8·32)

(iv) When x is negative and numerically less than $3a$, y^2 is negative and so the curve does not exist between $x = 0$ and $x = -3a$

When $x = -3a$, the expression for y becomes meaningless; but if $x \to -3a$ from the left, $y^2 \to +\infty$. In fact $x = -3a$ is an asymptote.

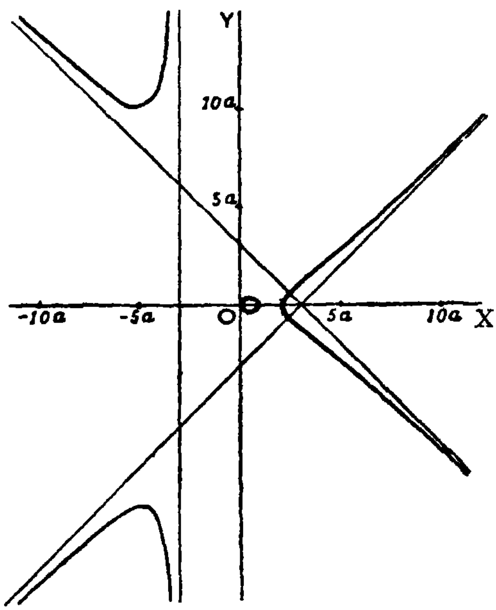
As $x \to -\infty$, $y^2 \to +\infty$. For numerically large values of x and y, we have (taking the negative value of y/x in (1) above)

$$y = -x + 3a - \frac{11}{2}\frac{a^2}{x} + \ldots$$

Hence the asymptote is $y = -x + 3a$, and in the 2nd quadrant also the curve is *above* the asymptote.

Thus in the second quadrant y is very large near $x = -3a$ and again very large for large negative values of x.

[For the intermediate values we can do one of two things. Either find y for a few values of x, say $x = -4a$, $-5a$ and $-6a$; or find where the tangent is horizontal (i.e., y has the least value. See Chapter XIII). For the latter we shall have to put $dy/dx = 0$ and solve. It is easy to see that we shall get thus a cubic in x, which might be difficult to solve. Hence we adopt the former alternative.]



When $\qquad x = -4a,\ y^2 = 120a^2,$

$$x = -5a,\ y^2 = 105a^2,$$
$$x = -6a,\ y^2 = 112a^2,$$
$$x = -7a,\ y^2 = 126a^2.$$

It is easy to see that y is a minimum somewhere near $x = -5a$; also, that as we take still larger negative values of x, y^2 would be larger.

Taking all the above facts into consideration, we see that the complete curve must be as shown in the figure.

[The curve appears to have two points of inflexion. By equating d^2y/dx^2 to zero, and solving for x, we can locate these if we like.]

Ex. 5. Trace the curve $x^5 + y^5 - 5a^2x^2y = 0$.

Here the equation is a quintic in y which cannot be solved. Neither can we solve the given equation for x. But transforming the equation to polars by putting $x = r\cos\theta$, and $y = r\sin\theta$, we have at once

$$r^2 = \frac{5a^2\cos^2\theta\,\sin\theta}{\cos^5\theta + \sin^5\theta},$$

which can be easily traced by the method for polar curves (see below).

EXAMPLES

Trace the following curves:

1. (i) $y = x^3$, (ii) $y^2 = x^3$. 2. $y = x(x^2 - 1)$.

3. $y = x(x^2 + 1)$. 4. $y = (x - 2)(x + 1)^2$.

5. $x^2y = x + 1$. 6. $x^3y = x + 1$.

7. $y = x^3 - 7x + 6$. 8. $y = x^4 - 3x^2 + 2$.

9. $y(x^2 + 4a^2) = 8a^3$. [*Agra*, 1932]

10. $y = (x^2 + 1)/(x^2 - 1)$. [*Dacca*, 1937]

11. $y^3 = 2ax^2 - x^3$. [*I. C. S.*, 1934]

12. (i) $ay^2 = x^2(a - x)$, (ii) $ay^2 = x^2(x - a)$. [*Delhi*, 1935]

13. $9ay^2 = x(x - 3a)^2$. [*Punjab*, 1932]

14. $a^{3/2}y = (x - a)^2\sqrt{(x - b)}$, $a > b$. [*Nagpur*, 1925]

15. $x^2y^2 = a^2(x^2 - y^2)$. [*Bombay*, 1937]

16. $y^2(a^2 + x^2) = x^2(a^2 - x^2)$. [*Punjab*, 1935]

17. $y^2(x^2 + y^2) + a^2(x^2 - y^2) = 0$. [*Agra*, 1935]

18. $y^2(a + x) = (a - x)x^2$. [*I. C. S.* 1928]

19. $x^2y^2 = (1 + y)^2(4 - y^2)$. $x = \pm\frac{(1+y)\sqrt{4-y^2}}{y}$ [*Agra*, 1937]

20. $x^2 (y + 1) = y^2 (x - 4)$. [*Punjab*, 1934]

21. $x (y - x)^2 = ay^2$. $y = \{x^2 \pm \sqrt{4x^3}\}/(x-a)$ [*Allahabad*, 1932]

22. $a^5y^2 + 2a^3x^3y + x^7 = 0$.

23. $y = x^2/a + \sqrt{(a^2 - x^2)}$.

24. $y^4 + 2axy^2 = ax^3 + x^4$.

25. Show that the curve $(a^2 + x^2) y = a^2x$ has three points of inflexion, and trace it. [*Nagpur*, 1927]

26. Trace $y^2(x^2 - a^2) = x^2(x^2 - 4a^2)$, and mark all its asymptotes. [*Agra*, 1928]

10·4. Polar Equations.

1. Solve the equation for r and consider how r varies as θ increases from 0 (or some convenient value θ_1) to $+\infty$, and also as θ diminishes from 0 (or θ_1) to $-\infty$. This is sufficient to trace the curve. If necessary, or convenient, *form a table of corresponding values of θ and r.*

2. In most of the polar equations only periodic functions ($\sin \theta$, $\cos \theta$, etc.) occur, and so values of θ from 0 to 2π (or sometimes some multiple or submultiple of 2π) need alone be considered. The remaining values of θ give no new branches of the curve.

3. Again, if changing the sign of θ does not change the value of r, the curve is symmetrical about the initial line.

4. If the curve possesses an infinite branch, find the asymptote.

If $r \to \infty$ when $\theta \to \alpha$ (either from the left or the right), *it should not be assumed that $\theta = \alpha$ is an asymptote.* The asymptote might not exist at all; or even if it exists, it might be parallel to the radius vector $\theta = \alpha$. The asymptote must be found by an application of § 8·5.

Ex. 1. Trace the curve

$$r = a \sin 3\theta.$$

The following table gives corresponding values of θ and r/a.

$3\theta = 0$	Intermediate values	$\frac{1}{2}\pi$	Intermediate values	π	Intermediate values	$\frac{3}{2}\pi$	Intermediate values	2π
$\theta = 0$	,,	$\pi/6$	,,	$2\pi/6$	,,	$3\pi/6$	,,	$4\pi/6$
$r/a = 0$	positive and increasing	1	positive and decreasing	0	Negative and numerically increasing	-1	Negative and numerically diminishing	0.

It is evident that the greatest positive value of r is a. Hence the curve between $\theta = 0$ and $\theta = 120°$ is as shown in Fig. 1. As θ increases further, r/a goes through the same cycle of changes as before, being a maximum at $3\theta = 2\pi + \frac{1}{2}\pi$, i.e., $\theta = 150°$.

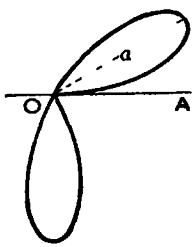


Fig. 1

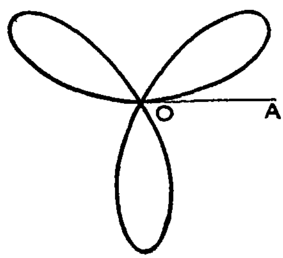


Fig. 2

If θ increases beyond π up to 2π, the same branches of the curve are repeated and we do not get any new branch. Hence the complete curve is as shown in Figure 2.

Values of θ outside the range $(0, 2\pi)$ need not be considered, as the value of r/a is periodic.

Ex. 2. Trace the curve $r^2\theta = a^2$.

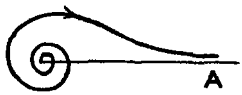
Consider first the positive value of r. When $\theta = 1$, i.e., 1 radian (or about $57°$), $r = a$. As θ increases, r continually diminishes. Thus the values of r when $\theta = 2\pi, 4\pi, 6\pi, \ldots$ are $0 \cdot 4a$, $0 \cdot 28a$, $0 \cdot 23a, \ldots$ As $\theta \to \infty$, $r \to 0$. So this part of the curve consists of a spiral in which the curve makes an indefinitely large number of revolutions round the pole, always coming nearer to it.

As θ diminishes from 1, r increases. As $\theta \to 0$ from positive values, $r \to \infty$.

The asymptote, by the method of § 8·5, is $\theta = 0$.

Hence the curve is as shown. This curve is called the Lituus.

To trace the curve for negative values of r, we notice that the point $(-r, \theta)$ is the same as the point $(r, \theta + \pi)$. Hence the branch of the curve which corresponds to $a/r = -|\sqrt{\theta}|$ can be obtained by turning the curve corresponding to the positive values of r through two right angles. For the sake of clearness this branch has not been drawn in the figure.



Ex. 3. Trace the curve

$$x^5 + y^5 - 5a^2x^2y = 0.$$

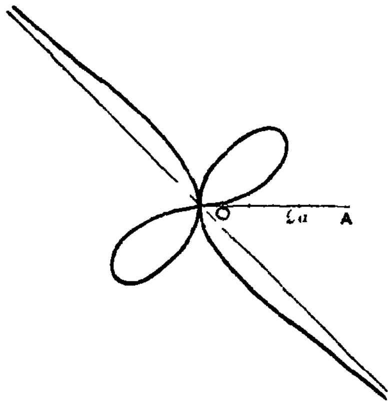
Transforming to polars, we have

$$r^2 = \frac{5a^2 \cos^2\theta \sin\theta}{\cos^5\theta + \sin^5\theta}.$$

Consider first the positive values of r.

When $\theta = 0$, $r = 0$.

r is again zero when $\theta = \frac{1}{2}\pi$. Between these values r is positive.



When θ increases beyond $\frac{1}{2}\pi$, but is less than $\frac{3}{4}\pi$, r^2 is positive. As θ goes on increasing and thus tends to $\frac{3}{4}\pi$, $r^2 \to \infty$.

For values of θ between $\frac{3}{4}\pi$ and π, r^2 is negative and so r is imaginary. Values of θ between π and 2π need not be considered because of symmetry in opposite quadrants, which is evident from the Cartesian equation.

The Cartesian equation shows also that the tangents at the origin are $x^2y = 0$. The asymptote can be found to be $x + y = 0$.

Hence the curve is as shown in the figure.

10·41. Special Methods.

1. In some cases, when the expression for y (or x, or r) consists of the sum or the difference of two terms, it is easier to draw the curve for each term separately and then draw the given curve as the curve whose

ordinates (or abscissae or radii vectors) are the sum or the difference of the corresponding quantities of the two curves previously drawn.

2. Similarly, if the expression for y, x, or r is a product of two factors, it is sometimes easier to draw first the curve corresponding to one factor, and then consider how the second factor modifies the curve.

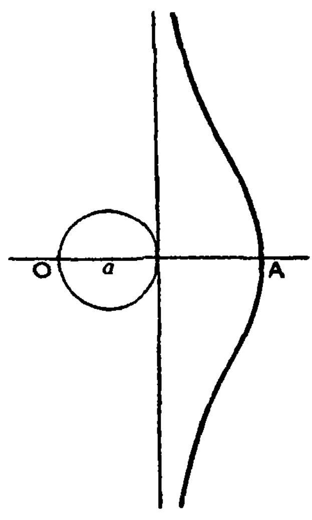
3. If the equation to a curve is given in a parametric form, $x = f(t), y = \varphi(t)$, the parameter t can in some cases be easily eliminated and the curve traced. But generally it is more convenient to give to t a series of values and plot the corresponding values of x and y, noting how x and y would behave (e.g., increase or decrease) for the intermediate values.

Ex. Trace the curve

$r = a(\sec\theta + \cos\theta)$. [*Allahabad*, 1927]

Here $r = a\cos\theta$ is the circle shown by the thin line in the figure; and $r = a\sec\theta$, i.e., $r\cos\theta = a$, is the straight line $x = a$.

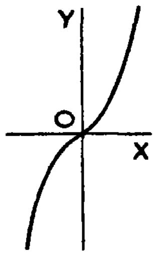
Hence, it is easy to draw the required curve, because its radius vector in every direction is the sum of the radii vectores of the two curves just drawn. The required curve is shown by the thick line.



10·5. Some well-known Curves.

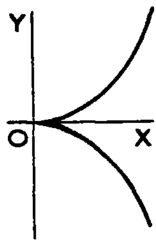
It is desirable that the student should be familiar with some of the well-known curves. Their equations and shapes are given below, but the student should first trace them independently and then compare his result with the figures given here. The student is supposed to be familiar with the conic sections, and the graphs of the circular, logarithmic and the exponential functions. Hence they are not given here.

Cubical Parabola*



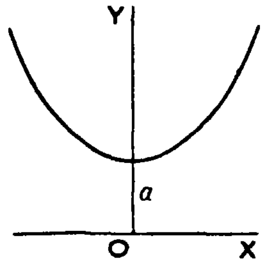
$a^2y = x^3$

Semi-Cubical Parabola



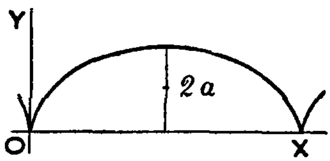
$ay^2 = x^3$

Catenary



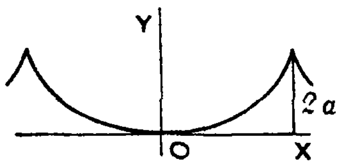
$y = a \cosh (x/a)$

Inverted Cycloid



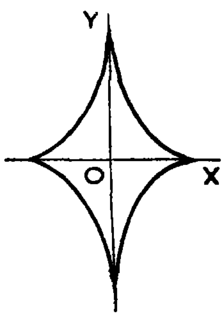
$$x = a\,(\theta - \sin \theta),$$
$$y = a\,(1 - \cos \theta)\,;$$
$$\text{whence } x = a \operatorname{vers}^{-1} (y/a) - \sqrt{(2ay - y^2)}.$$

Cycloid



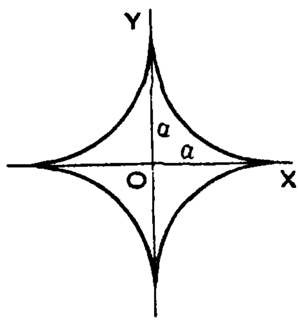
$$x = a\,(\theta + \sin \theta),$$
$$y = a\,(1 - \cos \theta)\,;$$
$$\text{or } x = a \operatorname{vers}^{-1} (y/a) + \sqrt{(2ay - y^2)}.$$

Evolute of an Ellipse



$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}$$

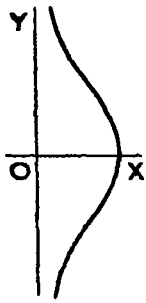
Astroid



$$x^{2/3} + y^{2/3} = a^{2/3}$$
$$\text{or} \quad x = a \cos^3\theta,\; y = a \sin^3\theta$$

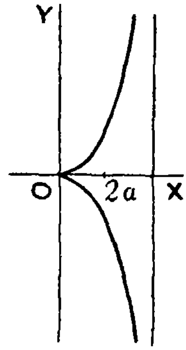
*Most of these curves have been drawn after Granville: *Differential and Integral Calculus.*

Witch of Agnesi[1]



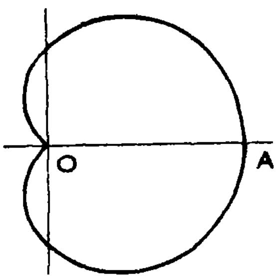
$xy^2 = 4a^2(2a - x)$

Cissoid of Diocles[2]



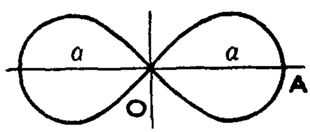
$y^2(2a - x) = x^3$

Cardioid



$r = a(1 + \cos\theta)$

Lemniscate of Bernoulli[3]



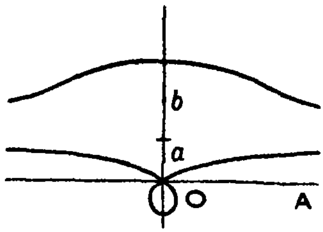
$r^2 = a^2 \cos 2\theta$

[1] Called after the Italian lady mathematician Maria Gaetna Agnesi (1718-1799) who treated it in her book *Instituzioni Analitiche*. "Agnesi was a somnambulist. Several times it happened to her that she went to her study, while in the somnambulist state, made a light, and solved some problem she had left incomplete when awake. In the morning she was surprised to find the solution carefully worked out on paper"—Cajori, *A History of Mathematics*.

[2] Diocles was a Greek mathematician (flourished about 180 B.C.). This curve he used for finding two mean proportionals between two given straight lines.

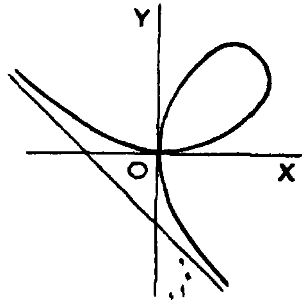
[3] Called after Jakob Bernoulli (1654-1705), professor of mathematics at the University of Basel (Switzerland). He and his brother Johann Bernoulli were staunch friends of Leibnitz and they enriched the calculus immensely in its early days (see Historical Note at the end of this book). The lemniscate was first considered by Jakob Bernoulli in the *Acta eruditorum* (1694). He also considered the

Conchoid of Nicomedes[4]



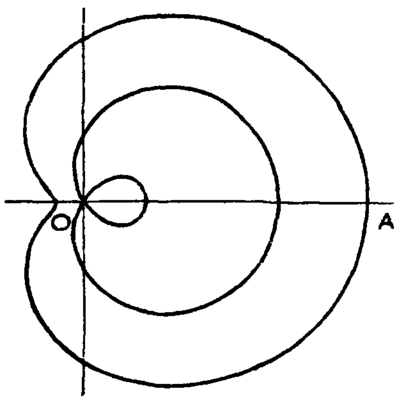
$r = a \operatorname{cosec} \theta \pm b$

Folium of Descartes[5]



$x^3 + y^3 - 3axy = 0$

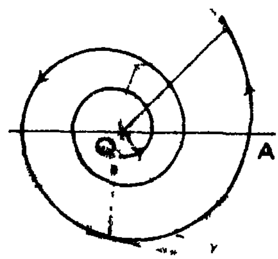
Limaçon



$r = a + b \cos \theta$

The two curves shown in the figure correspond to $a < b$ and $a > b$ respectively. The Cardioid is only a particular case of this.

The Logarithmic or the Equiangular Spiral.



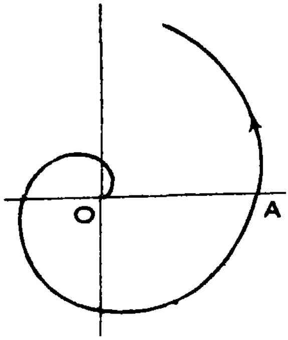
$r = ae^{m\theta}$

logarithmic spiral, which so fascinated him that he willed the curve to be engraved upon his tombstone.

[4] Greek mathematician. Nicomedes and Diocles were contemporaries (about 180 B.C.). Nicomedes devised a little machine by which the conchoid could be easily drawn. He used this curve to trisect angles and construct cubes whose volumes were double the volumes of given cubes.

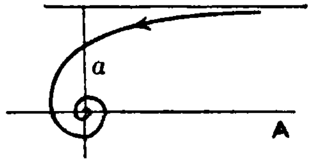
[5] René Descartes (1596-1650) was the great French mathematician and philosopher after whom the Cartesian axes are named. He

Spiral of Archimedes[6]



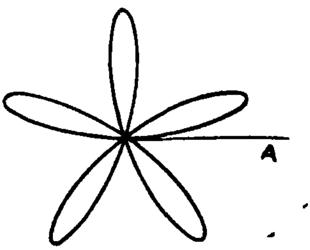
$r = a\theta$

Reciprocal Spiral



$r\theta = a$

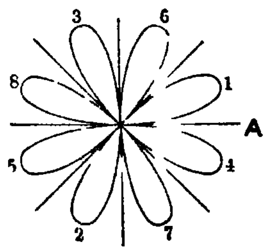
$2m + 1$ leaved rose



$r = a \sin (2m + 1)\theta$

The case $2m + 1 = 5$ is illustrated in the figure

$4m$ leaved rose



$r = a \sin 2m\theta$

The case $2m = 4$ is illustrated in the figure. Notice that in the curve $r = a \sin n\theta$, *if n is odd, there are only n loops*, but if n is even, there are $2n$ loops. The order in which the loops are described is indicated in the figure by numbers.

made many important discoveries It was in an attack on Fermat's method of finding tangents to curves that he gave the curve $x^3 + y^3 = axy$ as a curve to which Fermat's method would not apply. The text was accompanied by a figure which shows that Descartes did not know the correct shape of this curve. The correct shape was first given by C. Huygens (famous Dutch scientist) 54 years later.

[6] Archimedes (287 ?—212 B.C.) of Syracuse (Greece) was the greatest mathematician of antiquity. He wrote on Geometry and Mechanics, and made many fundamental discoveries of the utmost

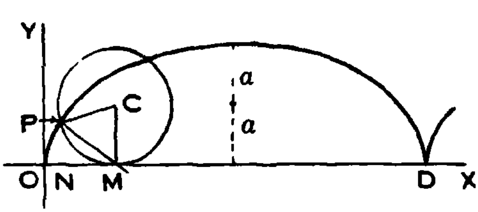
10·51. Properties of some well-known curves. (i) **Cycloid.** *The cycloid is the curve traced out by a point on the circumference of a circle which rolls (without sliding) along a straight line.*

Let OX be the straight line along which the generating circle MP rolls, and P the point on the circle which generates the curve.

Let the radius of the generating circle be a and its centre C.

Let P be the point (x, y) and let the $\angle PCM = \theta$

Then the arc $PM = a\theta$, and supposing P was in contact with O when the circle began rolling, $a\theta$ must be equal to OM. Hence, if N be the foot of the perpendicular from P on OX,



$$x = OM - NM = a\theta - PC \sin\theta = a(\theta - \sin\theta),$$

$$y = PN = MC - PC\cos\theta = a(1 - \cos\theta)$$

These two equations constitute the parametric equations of the cycloid.

The straight line on which the generating circle rolls is called the *base* of the cycloid.

(ii) **Catenary.** The catenary is the curve in which a heavy perfectly flexible string hangs when suspended by two points, as shown in books on Statics It is also called the *chainette*

(iii) **Equiangular Spiral.** The angle ϕ between the radius vector and the tangent is constant for this curve Hence the name

EXAMPLES

Trace the following curves:

1. $r = a\cos 3\theta$ 2. $r = a(\theta + \sin\theta)$.
3. $r = a\theta\sin\theta$. [Hint Trace $r = a\sin\theta$ first]
4. (i) $r^2 = a^2\cos 2\theta$, (ii) $x^3 + y^3 = 2ay^2$. [*Nagpur*, 1931]
5. $r^2\cos\theta = a^2\sin^3\theta$. 6. $r\theta = a\sin\theta$
7. $xy^2 + x^2y = a^3$. [*Lucknow*, 1932] 8. $r(1 + \theta) = a\theta$
9. $r(1 + \theta^2) = a\theta^2$. 10. $(r - a)^2 = 4ab\theta$

importance. His geometrical proofs were beautifully rigorous. By a special postulate, now known as the postulate of Archimedes, he excluded infinitesimals (see Chap. XIV), and his demonstrations were based on the strict theory of limits. However, his postulate did not commend itself to mathematicians until the modern arithmetical theory of limits was created (see Historical Note).

EXAMPLES ON CHAPTER X

1. Prove that the curve

$$ay^2 = (x-a)^2(x-b)$$

has, at $x = a$, a conjugate point if $a < b$, a node if $a > b$, and a cusp if $a = b$

2. In the curve

$$(y - x^2)^2 = x^n,$$

show that the origin is a cusp of the first or second species, according as n is $<$ or $>$ 4.

3. Show that the cardioid $r = a(1 + \cos\theta)$ has a cusp at the origin.

4. For the curve $y^2(a^2 + x^2) = x^2(a^2 - x^2)$ show that the origin is a node and that the nodal tangents bisect the angles made by the axes.

5. Show that the curve

$$x^5 + y^5 = 5ax^2y^2$$

has two cusps of the first species at the origin, and obtain its real asymptote. [*Patna*, 1935]

6. Find the double points and points of inflexion, if any, of the curve $a^2/x^2 - b^2/y^2 = 1$, and trace it. [*Nagpur*, 1928]

7. Trace the curve

$$r(\theta^2 - \cos\theta) = \theta^2 + \cos\theta,$$

and show that the circle of unit radius is an asymptotic circle.

8. Find the asymptotes of

$$(y - x)^2 x - 3y(y - x) + 2x = 0,$$

and trace the curve represented by the above equation. [*Agra*, 1934]

9. Find the position and nature of the double points on the curve $y^3 = x^3 + ax^2$.

Find also the asymptotes of the curve and trace it. [*Allahabad*, 1931]

10. Show that the curve

$$ay^2 = x^2y + x^3$$

has a cusp of the first species at the origin and an asymptote $x + y = a$ cutting the curve at $(\frac{1}{2}a, \frac{1}{2}a)$. Trace the curve. [*Allahabad*, 1930]

11. Examine the nature of the origin on the curve

$$y^2 = 2x^2y + x^4y - 2x^4.$$ [*Allahabad*, 1929]

12. Trace the curve

$$(x - y)^2(x + y)(2x + y) = a^2y^2.$$ [*Bombay*, 193[illegible]]

13. Trace the curve

$$a^4y^2 = x^5(2a - x). \qquad \text{[}Allahabad\text{, 1928]}$$

14. Examine the nature of the origin on the curve

$$ay^2 - x^3 + bx^2 = 0,$$

where a and b have opposite signs. [*Allahabad*, 1926]

15. Trace the curve $y = \frac{1}{2}x + x^{-1}$. [*Allahabad*, 1926]

16. Find the asymptotes of the curve

$$x^2y^2(x^2 - y^2)^2 = (x^2 + y^2)^2, \qquad \text{[}Benares\text{, 1930]}$$

and approximately trace the curve.

17. Sketch the form of the curve

$$y = ae^{-kx} \sin \alpha x$$

for positive values of x, and prove that it touches the curves $y = \pm\, a\,e^{-kx}$ each at an infinite number of points.

Also show that the successive maximum values of y form a series in geometrical progression, a, k and α being real quantities. [*I. C. S.*, 1932]

18. Trace the curve

$$y(y + 2x)(y - x)^2 = 9ax^3. \qquad \text{[}Madras\text{, 1937]}$$

19. Trace the curve $y^2 = (x - 2)^2(x - 5)$ and show that the line joining the points of inflexion subtends a right angle at the double point. [*Nagpur*, 1934]

20. Trace the curve $x = \cos 2t$, $y = \sin 3t$, for real values of t. Obtain the (x, y) equation of the curve and sketch its graph. Explain why the two graphs are not identical. [*Math. Tripos*, 1928]

21. Trace the curve

$$(x^2 - 1)\,y^2 = x,$$

determining its tangents parallel to the axes and points of inflexion, if any. [*Math. Tripos*, 1929]

22. Trace the curve

$$x^4 - y^4 = x^2y + x^2 - y^2. \qquad \text{[}Lucknow\text{, 1931]}$$

23. Trace the curve

$$r = a(2\cos\theta + \cos 3\theta). \qquad \text{[}Allahabad\text{, 1934]}$$

24. Sketch the curve whose equation in polar coordinates is

$$r^2(a^2 + b^2\tan^2\tfrac{1}{2}\theta) = a^4,$$

where

$$a > b > 0. \qquad \text{[}Math.\ Tripos\text{, 1932]}$$

CHAPTER XI

PARTIAL DIFFERENTIATION

11·1. Definitions. Let z be a symbol which has one definite value for every pair of values of x and y. Then z is called a *function of the two variables x and y.*

A similar definition can be given for functions of more than two variables.

z would be called a function of x and y even if z is defined only for every pair of values of x and y such that the point (x, y) lies within a certain area in the x, y plane. This area would then be called the *domain* of x and y. *Multiple-valued functions* may be regarded as defining two or more functions, as in the case of functions of a single variable.

A function of x and y is also written as $f(x, y)$, or $\phi(x, y)$, etc.

11·11. Continuity. $f(x, y)$ is said to be continuous at a point (a, b) if, for any arbitrarily chosen positive number ϵ, however small (but not zero), we can find a corresponding number δ such that

$$|f(x, y) - f(a, b)| < \epsilon,$$

for every point (x, y) within the circle with its centre at (a, b) and radius δ.

It must not be supposed that $\lim_{x \to a} \{\lim_{y \to b} f(x, y)\}$ is necessarily equal to $\lim_{y \to b} \{\lim_{x \to a} f(x, y)\}$. Thus

$$\lim_{x \to 0}\left\{\lim_{y \to 0} \frac{y - x}{y + x}\right\} = \lim_{x \to 0} \frac{-x}{x} = -1.$$

$$\text{But } \lim_{y \to 0}\left\{\lim_{x \to 0} \frac{y - x}{y + x}\right\} = \lim_{y \to 0} \frac{y}{y} = +1.$$

Hence the two limits are not the same in this case. Usually, however, they are the same.

Throughout this chapter we shall suppose that all the functions considered are continuous and their partial differential coefficients, as defined below, exist.

11·12. Partial differential coefficients. The partial differential coefficient of $f(x, y)$ with respect to x is the

ordinary differential coefficient of $f(x, y)$ when y is regarded as a constant. It is written as

$$\frac{\partial f}{\partial x} \text{ or } \partial f/\partial x.$$

Thus $$\frac{\partial f}{\partial x} = \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h}.$$

Again, the partial differential coefficient $\partial f/\partial y$ of $f(x, y)$ with respect to y is the ordinary differential coefficient of $f(x, y)$ when x is regarded as a constant.

Thus $$\frac{\partial f}{\partial y} = \lim_{k\to 0} \frac{f(x, y+k) - f(x, y)}{k}.$$

Similarly, if f is a function of the n variables x_1, x_2, ... x_n, the partial differential coefficient of f with respect to x_1 is the ordinary differential coefficient of f when all the variables except x_1 are regarded as constants, and is written as $\partial f/\partial x_1$.

$\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are also denoted by f_x and f_y respectively.

The partial differential coefficients of f_x and f_y are f_{xx}, f_{xy}, f_{yx}, f_{yy}; or

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y^2},$$

respectively. It should be specially noted that

$$\frac{\partial^2 f}{\partial y \partial x} \text{ means } \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right), \text{ and } \frac{\partial^2 f}{\partial x \partial y} \text{ means } \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right).$$

The student will be able to convince himself that in all ordinary cases

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}.$$

We shall for the present assume this proposition. A proof will be given Chapter XV.

Ex. Find the partial differential coefficients of $x^4 + x^2y^2 + y^4$.

If $f(x, y)$ denotes $x^4 + x^2y^2 + y^4$, we have

$$\frac{\partial f}{\partial x} = 4x^3 + 2xy^2, \qquad \frac{\partial f}{\partial y} = 2x^2y + 4y^3,$$

$$\frac{\partial^2 f}{\partial y\partial x} = 4xy, \qquad \frac{\partial^2 f}{\partial x\partial y} = 4xy,$$

$$\frac{\partial^2 f}{\partial x^2} = 12x^2 + 2y^2, \qquad \frac{\partial^2 f}{\partial y^2} = 2x^2 + 12y^2, \text{ etc.}$$

Notice that in the above $\dfrac{\partial^2 f}{\partial y\partial x} = \dfrac{\partial^2 f}{\partial x\partial y}$.

EXAMPLES

1. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ when

(i) $f = \tan^{-1}\dfrac{x^2 - y^2}{x - y}$, (ii) $f = \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - 1$,

(iii) $f = x^y$, (iv) $f = \log(x^2 + y^2)$.

2. Illustrate the theorem that

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{\partial^2 u}{\partial y\partial x},$$

when u is equal to

(i) $ax^2 + 2hxy + by^2$, (ii) $x\sin y + y\sin x$,

(iii) $x\log y$, (iv) $\log\{(x^2 + y^2)/xy\}$, [*Andhra*, 1937]

(v) $\log\tan(y/x)$, (vi) $(ay - bx)/(by - ax)$.

3. If $u = e^{xyz}$, show that

$$\frac{\partial^3 u}{\partial x\partial y\partial z} = (1 + 3xyz + x^2y^2z^2)\, e^{xyz}.$$

4. If $u = \tan^{-1}\dfrac{xy}{\sqrt{(1 + x^2 + y^2)}}$, show that

$$\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{(1 + x^2 + y^2)^{3/2}}.$$

5. If $x^x y^y z^z = c$, show that at $x = y = z$,

$$\frac{\partial^2 z}{\partial x\partial y} = -(x\log ex)^{-1}.$$

6. If $u = \sin^{-1} \dfrac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}$, show that

$$\frac{\partial u}{\partial x} = -\frac{y}{x}\frac{\partial u}{\partial y}. \qquad \text{[\textit{Lucknow}, 1932]}$$

If $z = f(x + ay) + \phi(x - ay)$, prove that

$$\frac{\partial^2 z}{\partial y^2} = a^2 \frac{\partial^2 z}{\partial x^2}. \qquad \text{[\textit{Nagpur}, 1933]}$$

8. If $1/u = \sqrt{(x^2 + y^2 + z^2)}$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0. \qquad \text{[\textit{Madras}, 1937]}$$

9. If $z(x + y) = x^2 + y^2$, show that

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right). \qquad \text{[\textit{Allahabad}, 1925]}$$

10. If $u = f(y/x)$, show that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0. \qquad \text{[\textit{Annamalai}, 1930]}$$

11·13. Total Differential Coefficient. If

$$u = f(x, y),$$

where $$x = \varphi(t), \text{ and } y = \psi(t),$$

then we can find the value of u in terms of t by substituting from the last two equations in the first equation. Hence we can regard u as a function of the single variable t, and find the ordinary differential coefficient du/dt.

Then du/dt is called the *total differential coefficient of u*, to distinguish it from the partial differential coefficients $\partial u/\partial x$ and $\partial u/\partial y$.

The problem now is to find du/dt without actually substituting the values of x and y in $f(x, y)$. We can obtain the requisite formula as follows :

Let $$\varphi(t + \tau) = x + h, \qquad \psi(t + \tau) = y + k,$$

Then, by definition,

$$du/dt = \lim_{\tau \to 0} \frac{f(x+h, y+k) - f(x, y)}{\tau}$$

$$= \lim_{\tau \to 0} \left\{ \frac{f(x+h, y+k) - f(x, y+k)}{h} \cdot \frac{h}{\tau} + \frac{f(x, y+k) - f(x, y)}{k} \cdot \frac{k}{\tau} \right\}$$

Now $\quad \lim_{\tau \to 0} \frac{h}{\tau} = \frac{dx}{dt}$, and $\lim_{\tau \to 0} \frac{k}{\tau} = \frac{dy}{dt}$.

Also, if k did not depend on h,

$$\lim_{h \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h}$$

would have been equal to

$$\frac{\partial f(x, y+k)}{\partial x},$$

by definition.

Moreover, supposing that $\partial f(x, y)/\partial x$ is a continuous function of y,

$$\lim_{k \to 0} \frac{\partial f(x, y+k)}{\partial x} = \frac{\partial f(x, y)}{\partial x}.$$

We shall *assume**, therefore, that

$$\lim_{\tau \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} = \frac{\partial f(x, y)}{\partial x}.$$

Hence $\quad \frac{du}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt};$

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}} \cdot \frac{\mathbf{dx}}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{y}} \cdot \frac{\mathbf{dy}}{\mathbf{dt}}.$$

Similarly, if $u = f(x_1, x_2, \ldots x_n)$, and $x_1, x_2 \ldots x_n$, are all functions of t, we can prove that

$$\frac{\mathbf{du}}{\mathbf{dt}} = \frac{\partial \mathbf{u}}{\partial \mathbf{x}_1} \cdot \frac{\mathbf{dx}_1}{\mathbf{dt}} + \frac{\partial \mathbf{u}}{\partial \mathbf{x}_2} \cdot \frac{\mathbf{dx}_2}{\mathbf{dt}} + \ldots + \frac{\partial \mathbf{u}}{\partial \mathbf{x}_n} \cdot \frac{\mathbf{dx}_n}{\mathbf{dt}}.$$

*The statement really requires proof. This will be given later; Chapter XV.

Ex. If $u = x^4y^5$, where $x = t^2$ and $y = t^3$, find du/dt.

We have, $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$
$$= 4x^3y^5 . 2t + 5x^4y^4 . 3t^2.$$

As a verification we notice that this value of du/dt
$$= 4(t^2)^3 (t^3)^5 . 2t + 5(t^2)^4 (t^3)^4 . 3t^2 = 23t^{22}.$$

Also, substituting the values of x and y in u, we have
$$u = (t^2)^4 (t^3)^5 = t^{23}.$$

Therefore $du/dt = 23t^{22}$, as before.

11·14. An important case. By supposing t to be the same as x in the formula for two variables in the last article, we get the following proposition:

When $f(x, y)$ is a function of x and y, and y is a function of x, the total (i.e., the ordinary) differential coefficient of f with respect to x is given by
$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx}.$$

Now, if we have an implicit relation between x and y of the form
$$f(x, y) = c,$$
where c is a constant and y is a function of x, the above formula becomes
$$0 = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\cdot\frac{dy}{dx},$$
which gives the important formula
$$\mathbf{\frac{dy}{dx} = -\frac{\partial f}{\partial x} \Big/ \frac{\partial f}{\partial y}}.$$

Again, *if f is a function of the n variables $x_1, x_2, x_3, \ldots x_n$, and $x_2, x_3, \ldots x_n$ are all functions of x_1, the total (i.e., the ordinary) differential coefficient of f with respect to x_1 is given by*
$$\mathbf{\frac{df}{dx_1} = \frac{\partial f}{\partial x_1} + \frac{\partial f}{\partial x_2}\cdot\frac{dx_2}{dx_1} + \frac{\partial f}{\partial x_3}\cdot\frac{dx_3}{dx_1} + \ldots + \frac{\partial f}{\partial x_n}\cdot\frac{dx_n}{dx_1}}.$$

Ex. 1. If $x^3 + 3x^2y + 6xy^2 + y^3 = 1$, find dy/dx.

If the left-hand side of the equation be denoted by f,

$$\partial f/\partial x = 3x^2 + 6xy + 6y^2,\ \partial f/\partial y = 3x^2 + 12xy + 3y^2.$$

Hence
$$\frac{dy}{dx} = -\frac{x^2 + 2xy + 2y^2}{x^2 + 4xy + y^2}.$$

We would evidently arrive at the same result by § 3·26.

Ex. 2. If $u = x^2 - y^2 + \sin yz$, where $y = e^x$, and $z = \log x$, find du/dx.

By the above formula,

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dx}.$$

$$= 2x + (-2y + z\cos yz)e^x + (y\cos yz)/x.$$

It is easy to verify that we shall get the same value of du/dx if we first convert u into a function of x alone by substituting in it the values of y and z, and then differentiate.

Examples

1. Find dy/dx if (i) $ax^2 + 2hxy + by^2 = 1$, (ii) $y^x + x^y = c$.

2. If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$, find du/dx.

3. Find du/dx if $u = \sin(x^2 + y^2)$, where $a^2x^2 + b^2y^2 = c^2$.

4. Find the partial differential coefficients of x^2y with respect to x and y, and its total differential coefficient with respect to x when x and y are connected by the relation $x^2 + xy + y^2 = 1$.

5. Prove that the tangent to the conic $3x^2 + 2xy - y^2 + 2x + 4y - 1 = 0$ is parallel to the axis of x at the points where it intersects the line $3x + y + 1 = 0$.

6. If $f(x, y) = 0$, $\phi(y, z) = 0$, show that

$$\frac{\partial f}{\partial y}\cdot\frac{\partial \phi}{\partial z}\cdot\frac{dz}{dx} = \frac{\partial f}{\partial x}\cdot\frac{\partial \phi}{\partial y}.$$ [*Punjab*, 1937]

11·2. Homogeneous Functions.

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots + a_{n-1}xy^{n-1} + a_ny^n$$

is an expression in which every term is of degree n. It is called a homogeneous function of degree n. This can be written also as

$$x^n\left\{a_0 + a_1\frac{y}{x} + a_2\left(\frac{y}{x}\right)^2 + \dots + a_n\left(\frac{y}{x}\right)^n\right\}.$$

Hence the general definition of a homogeneous function is as follows:

$x^n f(y/x)$ is called a *homogeneous function* of degree n, whatever the function f may be.

Thus $x^2 \sin (y/x)$ is a homogeneous function of x and y of degree 2.

Generally, if the function $f(x_1, x_2, \dots x_n)$ of the n variables $x_1, x_2, \dots x_n$ can be expressed in the form

$$x_r^n f\left(\frac{x_1}{x_r}, \frac{x_2}{x_r}, \dots \frac{x_n}{x_r}\right),$$

then $f(x_1, x_2, \dots x_n)$ is called a homogeneous function of $x_1, x_2, \dots x_n$ of degree n.

11·21. Euler's Theorem on Homogeneous Functions. *If $f(x, y)$ be a homogeneous function of x and y of degree n*, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf.$$

Let $f(x, y)$ be expressed in the form $x^n F(v)$ where v stands for y/x. Denoting dF/dv by F', we have

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\{x^n F(v)\} = n x^{n-1} F(v) + x^n F'(v) \cdot \frac{-y}{x^2},$$

and $$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\{x^n F(v)\} = x^n F'(v) \cdot \frac{1}{x}.$$

Hence $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = n x^n F(v) = n f(x, y).$$

In general, *if $f(x_1, x_2, \dots x_n)$ be a homogeneous function of degree n, then*

$$x_1\frac{\partial f}{\partial x_1} + x_2\frac{\partial f}{\partial x_2} + \dots + x_n\frac{\partial f}{\partial x_n} = nf.$$

The proof is similar to that for two variables. Thus if

$$f(x_1, x_2, \dots x_n) = x_1^n F(v, w, \dots)$$

where $v = x_2/x_1$, $w = x_3/x_1$, etc., then

$$\frac{\partial f}{\partial x_1} = n x_1^{n-1} F(v, w, \dots) + x_1^n\left\{\frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x_1} + \frac{\partial F}{\partial w} \cdot \frac{\partial w}{\partial x_1} + \dots\right\}.$$

But $$\frac{\partial v}{\partial x_1} = -\frac{x_2}{x_1^2}, \text{ etc.}$$

Hence $$\frac{\partial f}{\partial x_1} = nx_1^{n-1} F - x_1^{n-2} \left\{ x_2 \frac{\partial F}{\partial v} + \dots \right\}.$$

Again $$\frac{\partial f}{\partial x_2} = x_1^n \frac{\partial F}{\partial v} \cdot \frac{\partial v}{\partial x_2} = x_1^{n-1} \frac{\partial F}{\partial v};$$

Multiplying these equations by x_1, x_2, ... and adding, we get the result.

Ex. Verify Euler's theorem when
$$f(x, y, z) = 3x^2yz + 5xy^2z + 4z^4.$$

Here $$\frac{\partial f}{\partial x} = 6xyz + 5y^2z.$$

Therefore $$x\frac{\partial f}{\partial x} = 6x^2yz + 5xy^2z.$$

Similarly $$y\frac{\partial f}{\partial y} = 3x^2yz + 10xy^2z,$$

and $$z\frac{\partial f}{\partial z} = 3x^2yz + 5xy^2z + 16z^4.$$

Hence $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} + z\frac{\partial f}{\partial z} = 12x^2yz + 20xy^2z + 16z^4$$
$$= 4f(x, y, z),$$
which verifies Euler's theorem in this case.

EXAMPLES

1. Verify Euler's Theorem in the following cases :

 (i) $f(x, y) = ax^2 + 2hxy + by^2$,

 (ii) $f(x, y, z) = axy + byz + czx$.

2. Verify Euler's Theorem for the function

 (i) $u = x(x^3 - y^3)/(x^3 + y^3)$. [*Patna*, 1937]

 (ii) $u = (x^{1/4} + y^{1/4})/(x^{1/5} + y^{1/5})$. [*Agra*, 1936]

3. If u be a homogeneous function of degree n, show that
$$x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (n-1)\frac{\partial u}{\partial x},$$
$$x\frac{\partial^2 u}{\partial x \partial y} + y\frac{\partial^2 u}{\partial y^2} = (n-1)\frac{\partial u}{\partial y}.$$

4. If $u = \sin^{-1}(x/y) + \tan^{-1}(y/x)$, prove that

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0.$$ [*Nagpur*, 1928]

5. From Euler's theorem:

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu,$$

where u is a homogeneous function of x and y of degree n, deduce that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u.$$

6. If $u = \sin^{-1}\{(x^2+y^2)/(x+y)\}$, show that

$$x\,\partial u/\partial x + y\,\partial u/\partial y = \tan u$$ [*Delhi*, 1936]

11·3. What variable is to be regarded as constant. Consider the following example:

The area z of a right-angled triangle in which the sides containing the right angle are x and y, is given by

$$z = \tfrac{1}{2}xy. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Also, if the hypotenuse be denoted by t, and the length of one side by x as before, then the other side $y = \sqrt{(t^2 - x^2)}$.

Hence also

$$z = \tfrac{1}{2}x(t^2 - x^2)^{1/2}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

From (1) $$\frac{\partial z}{\partial x} = \tfrac{1}{2}y. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

But from (2) $$\frac{\partial z}{\partial x} = \tfrac{1}{2}(t^2 - x^2)^{1/2} - \tfrac{1}{2}x^2(t^2 - x^2)^{-1/2}$$

$$= \tfrac{1}{2}(y - x^2/y). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

The expressions on the right in (3) and (4) are not the same, though both represent dz/dx.

Evidently, therefore, $\partial z/\partial x$ *depends also on the variable which is regarded as a constant when z is differentiated with respect to x.*

Whenever there is a possibility of doubt arising as to which variable is regarded as a constant when z is differentiated with respect to x, the former is indicated by a suffix. Thus

$$\left(\frac{\partial z}{\partial x}\right)_{y=\text{const}} = \tfrac{1}{2}y, \text{ and } \left(\frac{\partial z}{\partial x}\right)_{t=\text{const}} = \tfrac{1}{2}(y - x^2/y).$$

A similar proposition is true also when z is a function of more than two variables.

[If the student draws a right angled triangle having sides of 3", 4", and 5", and then draws two other right angled triangles, in each of which one side is now 3·1" instead of 3", but in one of the triangles the other side is the same as before (viz., 4") and in the other the hypotenuse is the same as before (viz. 5"), he will see at once that the increase in area (viz. δz) is different in the two cases. So $\delta z/\delta x$ is different. As $\lim_{\delta x \to 0} \delta z/\delta x = \partial z/\partial x$, the latter will evidently be different in the two cases.]

It is very often necessary to change from Cartesian coordinates to polars, and vice versa. It must be remembered that in this connection $\partial/\partial x$ always means $(\partial/\partial x)_{y=\text{const}}$, and $\partial/\partial y$ means $(\partial/\partial y)_{x=\text{const}}$. Similarly $\partial/\partial r$ means $(\partial/\partial r)_{\theta=\text{const}}$, and $\partial/\partial\theta$ means $(\partial/\partial\theta)_{r=\text{const}}$.

Thus $\qquad x = r\cos\theta$, so $\dfrac{\partial x}{\partial r} = \cos\theta$.

But, although $\qquad r = x\sec\theta$,

$$\frac{\partial r}{\partial x} \neq \sec\theta,$$

because this value of $\partial r/\partial x$ is really $(\partial r/\partial x)_{\theta=\text{const}}$, and we want $(\partial r/\partial x)_{y=\text{const}}$. In fact, since $r^2 = x^2 + y^2$, we have

$$2r\frac{\partial r}{\partial x} = 2x, \text{ i.e., } \frac{\partial r}{\partial x} = \frac{x}{r} = \cos\theta.$$

The simple rule is to express x and y in polars before finding the partial differential coefficients of x or y, and to express r and θ in Cartesian coordinates before finding the partial differential coefficients of r or θ.

11·31. The second differential coefficient of an implicit function. If $f(x, y) = 0$, we have seen that

$$\frac{dy}{dx} = -\frac{\partial f}{\partial x} \bigg/ \frac{\partial f}{\partial y}.$$

We can find d^2y/dx^2 by differentiating this.

For the sake of convenience

$$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \text{ and } \frac{\partial^2 f}{\partial y^2}$$

are often denoted by p, q, r, s, and t respectively. Adopting this notation, we have

$$\frac{dy}{dx} = -\frac{p}{q}.$$

Differentiating with respect of x, we have

$$\frac{d^2y}{dx^2} = -\frac{q\dfrac{dp}{dx} - p\dfrac{dq}{dx}}{q^2}. \quad . \quad . \quad . \quad . \quad . \qquad (1)$$

But $\dfrac{dp}{dx} = \dfrac{\partial p}{\partial x} + \dfrac{\partial p}{\partial y}\dfrac{dy}{dx}$ by § 11 14 $= r + s\left(-\dfrac{p}{q}\right) = \dfrac{qr - ps}{q}$.

Similarly $\quad \dfrac{dq}{dx} = \dfrac{\partial q}{\partial x} + \dfrac{\partial q}{\partial y}\dfrac{dy}{dx} = s + t\left(-\dfrac{p}{q}\right) = \dfrac{qs - pt}{q}$.

Substituting these values in (1), we get

$$\frac{d^2y}{dx^2} = -\frac{q^2r - 2pqs + p^2t}{q^3}.$$

Ex Find d^2y/dx^2 if $ax^2 + 2hxy + by^2 = 1$.

Here $\quad p = 2(ax + hy),\ q = 2(hx + by),$

$\quad r = 2a,\ s = 2h,\ t = 2b$

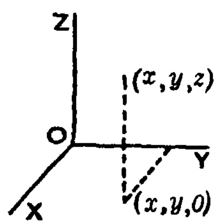
Hence

$$\frac{d^2y}{dx^2} = -\frac{(hx + by)^2 a - 2(ax + hy)(hx + by)h + (ax + hy)^2 b}{(hx + by)^3}.$$

11·4. Geometrical Meaning. If we take rectangular axes OX, OY, OZ, and suppose x, y, z to be the coordinates of a point referred to these axes, the equation

$$z = f(x, y) \quad . \quad . \quad . \qquad (1)$$

would represent a *surface* For, if at every point of the x, y plane we erect a perpendicular of length z given by equation (1), the other extremity of the perpendicular will evidently generate a surface.



The equation $y = c$, where c is a constant, will represent the plane parallel to the plane XOZ and at a distance c from it, because all points for which the value of y is c evidently lie on this plane.

Hence if we consider only those points of the surface $z = f(x, y)$ for which $y = c$, we shall get the curve in which the plane $y = c$ cuts the surface $z = f(x, y)$.

Let the plane $y = c$ cut OY in O', and take $O'X'$ to be parallel to OX. If P be a given point on the section of the surface $z = f(x, y)$ by the plane $y = c$ and Q any other point on it, it is clear

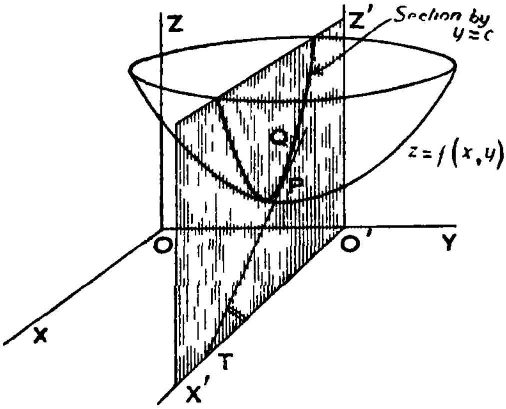
from the figure that as $Q \to P$, QP tends to the tangent at P to the section, and its inclination to $O'X'$ tends to

$$\lim_{h \to 0} \frac{f(x+h, c) - f(x, c)}{h},$$

i.e., to

$$\left(\frac{\partial f}{\partial x}\right)_{y=c}$$

Hence $\partial z/\partial x$ *at P is equal to the tangent of the angle which the tangent*



at P to the section of the surface $z = f(x, y)$, by a plane through P parallel to the plane XOZ, makes with a line drawn parallel to the axis of x.

A similar interpretation can be given to $\partial z/\partial y$.

11·5. Order of Partial Differentiations Commutative. With the help of the proposition

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \ (\S\ 11\cdot12),$$

it is easy to see that

$$\frac{\partial^3 u}{\partial x^2 \partial y} = \frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}u = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial x}u$$

$$= \frac{\partial}{\partial y}\frac{\partial}{\partial x}\frac{\partial}{\partial x}u = \frac{\partial^3 u}{\partial y \partial x^2}.$$

Similarly

$$\frac{\partial^{m+n} u}{\partial x^m \partial y^n} = \frac{\partial^{m+n} u}{\partial y^n \partial x^m},$$

and the proposition can be readily extended if there are more than two variables.

11·6. Taylor's Theorem for functions of two variables. By Taylor's theorem for functions of a single variable,

$$f(x+h, y+k) = f(x, y+k) + h\frac{\partial f(x, y+k)}{\partial x} + \frac{h^2}{2!}\frac{\partial^2 f(x, y+k)}{\partial x^2} + \dots$$

Expanding now each term,

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \frac{k^2}{2!}\frac{\partial^2 f(x, y)}{\partial y^2} + \dots \\ &\quad + h\frac{\partial}{\partial x}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \dots\right\} \\ &\quad + \frac{h^2}{2!}\frac{\partial^2}{\partial x^2}\left\{f(x, y) + k\frac{\partial f(x, y)}{\partial y} + \dots\right\} + \dots \end{aligned}$$

Hence

$$\begin{aligned} \mathbf{f(x+h, y+k)} &= \mathbf{f(x, y) + h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}} \\ &\quad \mathbf{+ \frac{1}{2!}\left(h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}\right) + \dots} \end{aligned}$$

This is sometimes written symbolically as

$$\begin{aligned} f(x+h, y+k) &= f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)f \\ &\quad + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \dots \\ &\quad + \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f + \dots \end{aligned}$$

Generally, if f is a function of the n variables $x_1, x_2, \dots x_n$, we shall have similarly

$$\begin{aligned} \mathbf{f(x_1+h_1, x_2+h_2, \dots, x_n+h_n)} &= \mathbf{f(x_1, x_2, \dots, x_n)} \\ &\quad \mathbf{+ \left(h_1\frac{\partial f}{\partial x_1} + h_2\frac{\partial f}{\partial x_2} + \dots + h_n\frac{\partial f}{\partial x_n}\right) + \dots} \end{aligned}$$

NOTE. $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$ means that we expand $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f$ as if h, ∂ and ∂x were algebraic quantities, but after the expansion is

obtained, $\frac{\partial f}{\partial x}, \frac{\partial^2 f}{\partial x^2}$, etc., are no longer considered as the products or quotients of powers of ∂ and ∂x multiplied by f. Thus $\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f$ means $h^2\frac{\partial^2 f}{\partial x^2} + 2hk\frac{\partial^2 f}{\partial x \partial y} + k^2\frac{\partial^2 f}{\partial y^2}$.

But all the laws of algebra cannot be applied in the symbolical method of writing. For, $\frac{\partial}{\partial x} f \neq f \frac{\partial}{\partial x}$, although $\frac{a}{bc} f = f \frac{a}{bc}$.

The consideration of the remainder after terms of degree n in h and k, and of the condition under which the infinite expansion is valid, is beyond the scope of this book.

11·61. Approximate Calculations. If x increases by δx, and y by δy, the new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$. Hence the increase $\delta f(x, y)$ in $f(x, y)$ will be equal to $f(x + \delta x, y + \delta y) - f(x, y)$.

Expanding the first term, and supposing that δx, δy are so small that their squares and higher powers can be neglected, we get

$$\delta f = \frac{\partial f}{\partial x}\delta x + \frac{\partial f}{\partial y}\delta y,$$

approximately.

This formula is very useful in calculating the effect of small errors in measured quantities.

If f is a function of n variables, we have similarly

$$\delta f = \frac{\partial f}{\partial x_1}\delta x_1 + \frac{\partial f}{\partial x_2}\delta x_2 + \dots + \frac{\partial f}{\partial x_n}\delta x_n \quad \textit{approximately.}$$

Ex. 1. Find the percentage error in the area of a rectangle when an error of $+1$ per cent. is made in measuring its length and breadth.

The area A of a rectangle $= xy$, where x and y are its sides.

Taking logs., $\log A = \log x + \log y$.

Now $\delta(\log A) = (1/A)\,\delta A$ by § 4·4

Hence, by Taylor's Theorem,

$$\frac{1}{A}\delta A = \frac{1}{x}\delta x + \frac{1}{y}\delta y.$$

Multiplying by 100, we have at once (since $100\,\delta x/x = 1$ and $100\,\delta y/y = 1$ by hypothesis),

the percentage error in $A = 2$.

Verification: The area of a rectangle with sides $x + x/100$, and $y + y/100$ is

$$\left(x + \frac{x}{100}\right)\left(y + \frac{y}{100}\right) = xy\left\{1 + \frac{2}{100} + \frac{1}{10000}\right\}$$

$$= xy\left(1 + \frac{2}{100}\right) \text{ nearly,}$$

and so, if the error in the sides is 1 per cent., the error in the area is approximately 2 per cent.

Ex. 2. The height h and the semi-vertical angle α of a cone are measured, and from them A, the total area of the cone, including the base, is calculated. If h and α are in error by small quantities δh and $\delta\alpha$ respectively, find the corresponding error in the area. Show further that, if $\alpha = \frac{1}{6}\pi$, an error of $+1$ per cent. in h will be approximately compensated by an error of $-0^\circ{\cdot}33$ in α.

[*Math. Tripos*, 1924]

Area of the base $= \pi h^2 \tan^2\alpha$, and area of the curved surface of the cone $= \frac{1}{2}.\ 2\pi h \tan\alpha.\ h\sec\alpha$

$$= \pi h^2 \tan\alpha \sec\alpha.$$

Therefore $\quad A = \pi h^2 (\tan^2\alpha + \tan\alpha\sec\alpha)$.

Therefore $\quad \delta A = \dfrac{\partial A}{\partial h}\delta h + \dfrac{\partial A}{\partial \alpha}\delta\alpha$ approximately.

$$= 2\pi h \tan\alpha(\tan\alpha + \sec\alpha)\,\delta h$$

$$+ \pi h^2(2\tan\alpha\sec^2\alpha + \sec^3\alpha + \tan^2\alpha\sec\alpha)\,\delta\alpha,$$

which gives the error δA in A corresponding to errors δh and $\delta\alpha$ in h and α respectively.

Putting $\alpha = \pi/6$, and $\delta h = h/100$, the above equation becomes

$$\delta A = 2\pi h^2 \times 0{\cdot}01 + \pi h^2 \times 3{\cdot}4646\,.\,\delta\alpha.$$

Hence $\quad \delta A = 0$ if $\delta\alpha = -\dfrac{2 \times 0{\cdot}01}{3{\cdot}4646}$ radian

$$= -\frac{2 \times 0{\cdot}01 \times 57{\cdot}3}{3{\cdot}4646} \text{ degree}$$

$$= -0^\circ{\cdot}33.$$

EXAMPLES

1. What is the approximate error in the volume and surface of a parallelopiped $10'' \times 20'' \times 30''$ if an error of $-0{\cdot}02$ inch is made in measuring each side?

2. The dimensions of a cone are : radius 4 inches, and altitude 6 inches. What is the error in volume and total surface (including the base) if there is a shortage of 0·01 in. per inch in the measure used?

3. In a triangle ABC the angles and the sides a and b are made to vary in such a way that the area remains constant; the side c also remains constant. Show that if a and b vary by small amounts δa and δb respectively, $\cos A.\ \delta a + \cos B.\ \delta b = 0$. [*Lucknow*, 1930]

4. The area of a triangle whose sides are a, b, c is $\triangle$. Prove that the error in the area corresponding to errors δa, δb, δc in the sides is approximately given by

$$2\triangle\delta\triangle = s^2\,\delta p - s\delta q - abc\,\delta s,$$

where

$$2s = a + b + c,$$
$$2p = a^2 + b^2 + c^2,$$
$$3q = a^3 + b^3 + c^3.$$

[*Math. Tripos*, 1924]

5. O is the centre of a circle and A, B are points on its circumference. It is required to measure the lesser of the two areas into which the chord AB divides the circle. With the instruments available it is known that angular measurements are liable to an error of $\pm\frac{1}{10}$th degree, and linear measurements to an error of $\pm$ 1 part in 500. If the area is calculated from the measurements of the angle AOB, which is found to be 45°, and the radius OA, show that the result is liable to be in error by $\pm$ 1 per cent., approximately. [*Math. Tripos*, 1926]

6. The sides of an acute-angled triangle ABC are measured. Prove that the increment in A due to small increments in a, b, c is given by the equation

$$bc\sin A.\ \delta A = -a\,(\cos C.\ \delta b + \cos.\ B\,\delta c - \delta a).$$

Supposing that the limits of error in the length of any side are $\pm\,\mu$ per cent., where μ is small, prove that the limits of error in A are approximately

$$\pm\,1\cdot15\,(\mu\, a^2/bc\sin A)\ \text{degrees.}$$ [*Math. Tripos*, 1927]

7. ABC is an acute-angled triangle with fixed base BC. If δb, δc, δA and δB are small increments in b, c, A and B respectively when the vertex A is given a small displacement δx parallel to BC, prove that

(i) $c\delta b + b\delta c + bc\cot A\ \delta A = 0$;

(ii) $c\delta B + \sin B\ \delta x = 0$. [*Math. Tripos*, 1930]

8. C is the middle point of an arc ACB of a circle. The radius of the circle is estimated from the measurements of the lengths x and y of the chords AB, AC respectively. If there is the same actual small error $+\,\alpha$ in each of the measurements of x and y, find the corresponding error in the calculated value of the radius and verify that, when the arc subtends an angle of 120° at the centre, this error in the radius is $-(2 - \sqrt{3})\alpha$. [*London*, 1934]

11·7. Change of variables. If

$$u = f(x, y), \quad \ldots \quad (1)$$

where $$x = \phi(t_1, t_2), \text{ and } y = \psi(t_1, t_2), \quad \ldots \quad (2)$$

it is frequently necessary to change expressions involving u, x, y, $\partial u/\partial x$, $\partial u/\partial y$, $\partial^2 u/\partial x^2$, etc., into expressions involving u, t_1, t_2, and the partial differential coefficients of u with respect to t_1, and t_2.

The necessary formulae are easily obtained. Regarding t_2 as a constant, we obtain, by § 11·13,

$$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_1}, \quad \ldots \quad (3)$$

where the ordinary differential coefficients in the formula of § 11·13 have been replaced by partial differential coefficients, because x and y are now functions of the two variables t_1 and t_2 instead of only one variable t.

Similarly, regarding t_1 as a constant, we get

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial t_2}. \quad \ldots \quad (4)$$

Equations (3) and (4), when solved as a pair of simultaneous linear equations, give us the values of $\partial u/\partial x$ and $\partial u/\partial y$ in terms of $\partial u/\partial t_1$, $\partial u/\partial t_2$, and the known quantities $\partial x/\partial t_1$, $\partial y/\partial t_1$, $\partial x/\partial t_2$, and $\partial y/\partial t_2$.

If we substitute these values of $\partial u/\partial x$, $\partial u/\partial y$ and the values of x and y as given by equations (2) in any expression involving u, x, y, $\partial u/\partial x$, $\partial u/\partial y$, we shall effect the required transformation.

In case equations (2) are easily solvable for t_1, and t_2, in terms of x and y, say

$$t_1 = F_1(x, y) \text{ and } t_2 = F_2(x, y). \quad \ldots \quad (5)$$

It is easier to use the formulae

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial x}, \\ \frac{\partial u}{\partial y} &= \frac{\partial u}{\partial t_1} \cdot \frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2} \cdot \frac{\partial t_2}{\partial y}, \end{aligned}\right\} \quad \ldots \quad (6)$$

in which the values of $\partial t_1/\partial x$, ... are to be substituted after finding them from (5).

The values of the higher differential coefficients of u can be found by a repeated application of the formulae (3) and (4), or of (6). The above formulae can be easily extended to the case of more than two independent variables.

A change from Cartesian coordinates (x, y) to polars (r, θ), where

$$x = r \cos \theta, \qquad y = r \sin \theta,$$

is very often required.

Ex. Transform the equation $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ into polar coordinates.

Here $\quad r=\sqrt{(x^2+y^2)}, \quad \theta=\tan^{-1}(y/x).$

Hence $\quad \frac{\partial r}{\partial x}=\frac{x}{\sqrt{(x^2+y^2)}}=\cos\theta, \quad \frac{\partial r}{\partial y}=\frac{y}{\sqrt{(x^2+y^2)}}=\sin\theta,$

$$\frac{\partial\theta}{\partial x}=-\frac{y}{x^2+y^2}=-\frac{\sin\theta}{r}, \quad \frac{\partial\theta}{\partial y}=\frac{x}{x^2+y^2}=\frac{\cos\theta}{r}.$$

Therefore, by (6), $\quad \frac{\partial u}{\partial x}=\frac{\partial u}{\partial r}\cos\theta+\frac{\partial u}{\partial\theta}\left(-\frac{\sin\theta}{r}\right),$

i.e., $$\frac{\partial u}{\partial x}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)u.$$

It follows that $\frac{\partial^2 u}{\partial x^2}=\left(\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right)\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$

$$=\cos\theta\frac{\partial}{\partial r}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial u}{\partial r}-\frac{\sin\theta}{r}\frac{\partial u}{\partial\theta}\right)$$

$$=\cos^2\theta\frac{\partial^2 u}{\partial r^2}-\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\sin^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$$
$$+\frac{\sin^2\theta}{r}\frac{\partial u}{\partial r}+\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}.$$

Similarly $\quad \frac{\partial^2 u}{\partial y^2}=\sin^2\theta\frac{\partial^2 u}{\partial r^2}+\frac{2\sin\theta\cos\theta}{r}\frac{\partial^2 u}{\partial r\partial\theta}+\frac{\cos^2\theta}{r^2}\frac{\partial^2 u}{\partial\theta^2}$
$$+\frac{\cos^2\theta}{r}\frac{\partial u}{\partial r}-\frac{2\sin\theta\cos\theta}{r^2}\frac{\partial u}{\partial\theta}.$$

Adding, we get

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}.$$

Hence the transformed equation is

$$\frac{\partial^2 u}{\partial r^2}+\frac{1}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2}=0.$$

Examples

1. If $u=f(r)$, where $r^2=x^2+y^2$, prove that

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=f''(r)+\frac{1}{r}f'(r).$$

2. If $u = f(y - z, z - x, x - y)$, prove that

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0.$$ [*Agra*, 1933]

3. Transform $$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

into polars and show that $v = \sin n\theta\,(Ar^n + Br^{-n})$ satisfies the above equation. [*Andhra*, 1936]

4. If $z = z(u, v)$, $u = x^2 - 2xy - y^2$, $v = y$, show that

$$(x + y)\frac{\partial z}{\partial x} + (x - y)\frac{\partial z}{\partial y} = 0$$

is equivalent to $\partial z/\partial v = 0$. [*Dacca*, 1936]

5. If $x = r\cos\theta$, $y = r\sin\theta$, $z = f(x, y)$, prove that

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial z}{\partial \theta}\sin\theta.$$

Prove also that

$$\frac{\partial^2 (r^n \cos n\theta)}{\partial x \partial y} = -n(n-1)r^{n-2}\sin(n-2)\theta.$$ [*Bombay*, 1935]

6. Prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2},$$

where $x = \xi\cos\alpha - \eta\sin\alpha$, $y = \xi\sin\alpha + \eta\cos\alpha$.

[*Bombay*, 1937]

EXAMPLES ON CHAPTER XI

1. (i) If $u = 2(ax + by)^2 - (x^2 + y^2)$, and $a^2 + b^2 = 1$, find the value of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}.$$

(ii) If $\theta = t^n e^{-r^2/4t}$, find what value of n will make

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \theta}{\partial r}\right) = \frac{\partial \theta}{\partial t}.$$ [*Allahabad*, 1934]

2. If $r^2 = x^2 + y^2$, prove that

$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} = \frac{1}{r}\left\{\left(\frac{\partial r}{\partial x}\right)^2 + \left(\frac{\partial r}{\partial y}\right)^2\right\}.$$ [*Punjab*, 1936]

3. If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, show that

$$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}.$$ [*Agra*, 1934]

4. If $u = x\,\phi(y/x) + \psi(y/x)$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0.$$

5. If $z = x^2 \tan^{-1}(y/x) - y^2 \tan^{-1}(x/y)$,

prove that $$\frac{\partial^2 z}{\partial x \partial y} = \frac{x^2 - y^2}{x^2 + y^2}.$$

6. Find d^2y/dx^2 if $x^4 + y^4 = 4a^2xy = 0$.

7. Find d^2y/dx^2 if $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$.

8. If $v = A\,t^{-1/2} e^{-x^2/4a^2t}$,

prove that $$\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}.$$

9. The equation

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$$

refers to the conduction of heat along a bar without radiation; show that if

$$u = A\,e^{-gx}\sin(nt - gx),$$

where A, g, n are positive constants, then

$$g = \sqrt{(n/2\mu)}.$$

10. If $$\frac{x^2}{a^2+u} + \frac{y^2}{b^2+u} + \frac{z^2}{c^2+u} = 1,$$

prove that $$u_x^2 + u_y^2 + u_z^2 = 2(xu_x + yu_y + zu_z).$$ [*Patna*, 1935]

11. Prove that the equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

is satisfied by $\phi = (Ar^n + Br^{-n})\cos n(\theta - \alpha)$.

12. If $\phi(x, y) = 0$ be the equation to a curve, prove that

$$\rho = \frac{\left\{\left(\dfrac{\partial\phi}{\partial x}\right)^2 + \left(\dfrac{\partial\phi}{\partial y}\right)^2\right\}^{\frac{3}{2}}}{\left(\dfrac{\partial\phi}{\partial y}\right)^2\dfrac{\partial^2\phi}{\partial x^2} - 2\dfrac{\partial\phi}{\partial x}\dfrac{\partial\phi}{\partial y}\dfrac{\partial^2\phi}{\partial x\partial y} + \left(\dfrac{\partial\phi}{\partial x}\right)^2\dfrac{\partial^2\phi}{\partial y^2}}.$$

13. If the curves $f(x, y) = 0$ and $\phi(x, y) = 0$ touch, show that at the point of contact

$$\frac{\partial f}{\partial x}\frac{\partial\phi}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial\phi}{\partial x} = 0.$$

14. If $z = 2xy^2 - 3x^2y$, and if x increases at the rate of 2 inches per second as it passes through the value $x = 3$ inches, show that if y is passing through the value $y = 1$ inch, y must decrease at the rate of $2\frac{2}{15}$ inches per second, in order that z shall remain constant.

[*Benares*, 1931]

15. The angles of a triangle are calculated from the sides a, b, c; if small changes δa, δb, δc are made in the sides, show that approximately

$$\delta A = \tfrac{1}{2}a\,(\delta a - \delta b \cos C - \delta c \cos B)/\Delta,$$

where Δ is the area of the triangle; and verify that

$$\delta A + \delta B + \delta C = 0. \qquad \text{[\textit{Benares}, 1926]}$$

16. The height of a tower is determined by observing the elevations θ and ϕ of its summit from two points in a direct line with the foot of the tower and at a distance a apart. Show that the error in the calculated height due to small errors $d\theta$ and $d\phi$ is approximately

$$a\,(\sin^2\theta\, d\phi - \sin^2\phi\, d\theta)\operatorname{cosec}^2(\theta - \phi) \qquad \text{[\textit{Andhra}, 1937]}$$

17. If x, y are the coordinates of any point and $x = r\cos\theta$, and $y = r\sin\theta$, find the values of

$$\left(\frac{\partial x}{\partial \theta}\right)_{r\ \text{const.}},\ \left(\frac{\partial \theta}{\partial x}\right)_{y\ \text{const.}}.$$

Verify the second result geometrically.

Prove that

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0. \qquad \text{[\textit{Math. Tripos}, 1927]}$$

18. A quantity y is derived from measurement of a quantity x by means of a relation $y = f(x, t)$, where t is a parameter connected with x by the relation $\phi(x, t) = 0$.

Show that if an error δx is made in the estimation of x, then the resulting error in the computed value of y is given approximately by the formula

$$\delta y = \frac{\dfrac{\partial f}{\partial x}\dfrac{\partial \phi}{\partial t} - \dfrac{\partial f}{\partial t}\dfrac{\partial \phi}{\partial x}}{\dfrac{\partial \phi}{\partial t}}\,\delta x.$$

The luminosity L of a star is connected with its mass M by the relation

$$L = aM\,(1 - \beta),$$

where β is a positive number less than unity connected with M by the relation

$$1 - \beta = b\beta^4M^2,$$

a and b being given constants. If p is the percentage error made in the estimate of M, express the resulting percentage error in the

calculated luminosity in terms of p and β, and show that it lies between p and $3p$. [*Math. Tripos*, 1923]

19. If $x = r\cos\theta$, $y = r\sin\theta$, show that

$$\frac{\partial r}{\partial x} = \frac{\partial x}{\partial r}; \quad \frac{\partial x}{r\,\partial\theta} = \frac{r\,\partial\theta}{\partial x} \quad \text{and} \quad \frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0.$$

[*Nagpur*, 1934]

20. In a plane triangle, if the angles and sides receive small variations, prove that

(*a*) $\cos C \cdot \delta a + \cos A \cdot \delta c = 0$; b, B being constants,

(*b*) $c \cdot \delta A + a\cos B \cdot \delta C = 0$; a, b being constants.

[*Patna*, 1932]

CHAPTER XII

ENVELOPES. EVOLUTES

12·1. Families of Curves. The equation

$$y = mx + \frac{1}{m}$$

represents, for a given value of m, a straight line. By giving different values to m, we get different straight lines. Suppose now that m continuously varies from one value to another. We get thus a "family" of straight lines.

In general, if $F(x, y, \alpha)$ is an expression involving x, y and α, the curves corresponding to the equation $F(x, y, \alpha) = 0$ constitute *a family of curves*. If α_1 is a particular value of α, the equation $F(x, y, \alpha_1) = 0$ represents one *member* of this family. The symbol α, which is constant for the same member, but different for different members of the family, is called the *parameter* of the family.

12·11. Envelope. Definition. A curve which touches each member of a family of curves, and at each point is touched by some member of the family, is called the *envelope* of that family of curves.

For instance, we know by Coordinate Geometry that all straight lines whose equation is of the form $y = mx + 1/m$ touch the parabola

$$y^2 = 4x,$$

and also that the parabola $y^2 = 4x$ has at every point a tangent which is of the form $y = mx + 1/m$. Hence we infer that the envelope of the family of straight lines

$$y = mx + 1/m$$

is the parabola $$y^2 = 4x.$$

12·12. Envelope. Another Definition. It will be shown presently that, in general, we can adopt the following working definition as equivalent to the previous one:

If $F(x, y, \alpha) = 0$ represents a family of curves whose parameter is α, and if the curves $F(x, y, \alpha) = 0$ and $F(x, y, \alpha + h) = 0$ cut in a point which tends to a definite point P as h tends to zero, the locus of P (for varying values of α) is called the envelope of the family.

12·13. Method of finding the Envelope. Let the given family of curves be

$$F(x, y, \alpha) = 0. \quad . \quad . \quad . \quad . \quad (1)$$

Supposing α to have a particular value, this equation represents one member of the family. Let another member of the family be

$$F(x, y, \alpha + h) = 0. \quad . \quad . \quad . \quad . \quad (2)$$

The coordinates of the point of intersection* P_1 of (1) and (2) will satisfy the equation

$$F(x, y, \alpha + h) - F(x, y, \alpha) = 0,$$

and, therefore, also the equation

$$\frac{F(x, y, \alpha + h) - F(x, y, \alpha)}{h} = 0.$$

Taking limits as $h \to 0$, we see that the coordinates of the point P to which P_1 tends as $h \to 0$ satisfy the equation

$$\frac{\partial F(x, y, \alpha)}{\partial \alpha} = 0. \quad . \quad . \quad . \quad . \quad (3)$$

Also the coordinates of P must satisfy (1), because P is a point on (1).

If we now eliminate α between (1) and (3), we shall get an equation which the coordinates of P will satisfy for all values of α. That is, the result of eliminating α between (1) and (3) will be the locus of P.

Hence, *the equation of the envelope of the family of curves* $F(x, y, \alpha) = 0$, *where* α *is the parameter, is obtained by eliminating* α *between the equations*

$$\mathbf{F(x, y, \alpha) = 0}$$

and

$$\frac{\partial \mathbf{F(x, y, \alpha)}}{\partial \alpha} = 0.$$

*The argument will apply even when there are more than one points of intersection.

Ex. 1. Find the envelope of the family of straight lines

$$y = mx + 1/m.$$

Differentiating partially with respect to m, we have

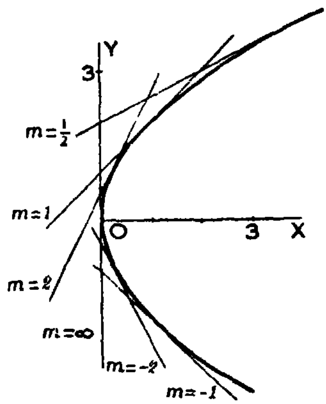
$$0 = x - 1/m^2, \text{ or } m = x^{-1/2}.$$

Substituting in the first equation, we have for the envelope the equation

$$y = x^{1/2} + x^{1/2} = 2x^{1/2},$$

or $$y^2 = 4x.$$

A study of the adjoined figure will be instructive in this connection.



Ex. 2. Find the envelope of the family of straight lines

$$\frac{ax}{\cos a} - \frac{by}{\sin a} = a^2 - b^2,$$

where the parameter is a.

Differentiating partially w. r. t. a, we have

$$\frac{ax \sin a}{\cos^2 a} + \frac{by \cos a}{\sin^2 a} = 0,$$

or $$\tan^3 a = -\frac{by}{ax}.$$

Therefore $$\tan a = -b^{1/3} y^{1/3}/a^{1/3} x^{1/3},$$

whence $$\sin a = \mp b^{1/3} y^{1/3}/(a^{2/3} x^{2/3} + b^{2/3} y^{2/3})^{1/2},$$

and $$\cos a = \pm a^{1/3} x^{1/3}/(a^{2/3} x^{2/3} + b^{2/3} y^{2/3})^{1/2}.$$

Substituting these values in the equation of the curve, we have

$$\pm (a^{2/3} x^{2/3} + b^{2/3} y^{2/3})^{1/2} \left(\frac{ax}{a^{1/3} x^{1/3}} + \frac{by}{b^{1/3} y^{1/3}}\right) = (a^2 - b^2),$$

or $$\pm (a^{2/3} x^{2/3} + b^{2/3} y^{2/3})^{3/2} = (a^2 - b^2),$$

i.e., $$a^{2/3} x^{2/3} + b^{2/3} y^{2/3} = (a^2 - b^2)^{2/3},$$

which is the equation of the required envelope.

12·14. Elimination in the case of a Quadratic. In case the equation $F(x, y, \alpha) = 0$ is merely a quadratic in α, say

$$A\alpha^2 + B\alpha + C = 0, \quad . \quad . \quad . \quad (1)$$

where A, B and C are functions of x and y, the result of differentiating partially with respect to α is

$$2A\alpha + B = 0.$$

Substituting the value of α from this in (1) we have

$$A\left(-\frac{B}{2A}\right)^2 + B\left(-\frac{B}{2A}\right) + C = 0,$$

or $$B^2 - 4AC = 0.$$

Hence *the envelope of the family of curves*

$$A\alpha^2 + B\alpha + C = 0,$$

where A, B, C are functions of x, is

$$\mathbf{B^2 - 4AC = 0.}$$

Ex. Find the envelope of the family of straight lines

$$y = mx + 1/m.$$

The equation of the family can be written as

$$m^2x - my + 1 = 0.$$

Hence the envelope is $(-y)^2 - 4 . x . 1 = 0$,

i.e., $$y^2 = 4x.$$

EXAMPLES

1. Find the envelope of the line $x \cos \alpha + y \sin \alpha = a$, the parameter being α, and interpret the result geometrically.

Find the envelope of the following family of straight lines :

2. $y = m^2x + 1/m^2$.

3. $y = mx + \sqrt{(a^2m^2 + b^2)}$, the parameter being m.

4. $y = mx + am^3$, the parameter being m.

5. $x/a + y/b = 1$, when $ab = c^2$, where c is a constant.

6. $y = mx + am^p$, where m is the parameter.

7. $x \cos m\theta + y \sin m\theta = a(\cos n\theta)^{m/n}$, where θ is the parameter.

8. Find the envelope of the family of circles

$$x^2 + y^2 - 2ax \cos \alpha - 2ay \sin \alpha = c^2,$$

where α is the parameter, and interpret the result.

Find the envelope of the following system of circles :

9. $(x - \alpha)^2 + y^2 = 4\alpha$.

10. $(x - \alpha)^2 + (y - \alpha)^2 = 2\alpha$.

Find the envelope of the following family of curves :

11. $tx^3 + t^2y = a$, the parameter being t.

12. $y = t^2(x - t)$.

13. Find the envelope of the straight lines

$$x \cos a + y \sin a = l \sin a \cos a,$$

where the parameter is the angle a. Give the geometrical interpretation. [*Bombay*, 1936]

12·2. The Envelope touches each member of the family. We shall now show that, *in general, the envelope of a family of curves, deduced on the basis of the second definition, touches each member of the family.*

Let any member of the family be

$$F(x, y, \alpha) = 0, \quad \ldots \ldots \quad (1)$$

where α is a constant and equal to α_1, say.

The equation of the envelope is the result of eliminating α between

$$\left.\begin{aligned} F(x, y, \alpha) &= 0, \\ \text{and} \quad \frac{\partial}{\partial \alpha} F(x, y, \alpha) &= 0. \end{aligned}\right\} \quad \ldots \ldots \quad (2)$$

Thus the equation of the envelope may be regarded as

$$F(x, y, \alpha) = 0, \quad \ldots \ldots \quad (3)$$

in whch α is not a constant, but a function of x and y given by

$$\partial F(x, y, \alpha) / \partial \alpha = 0. \quad \ldots \quad (4)$$

Consider now the point P on (1), where P is the limiting position to which the intersection of $F(x, y, \alpha_1) = 0$ and $F(x, y, \alpha_1 + h) = 0$ tends as $h \to 0$. This point P lies on the curve (1) and also on the envelope (2). The tangent at P to the curve (1) has the gradient dy/dx given by

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0, \quad \ldots \ldots \quad (5)$$

where, in the differentiations, α is kept constant and equal to α_1.

But the tangent at P to the envelope has the gradient dy/dx given by

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} + \left\{\frac{\partial F}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial x}\right\}_{\alpha = \alpha_1} = 0, \quad (6)$$

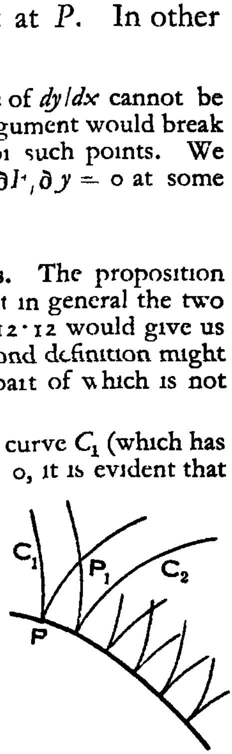
because a is not a constant for the envelope.

But in virtue of equation (4), which is satisfied at every point of the envelope, (6) reduces to (5); i.e., the gradients of the tangents to the curve and the envelope at the common point P are the same. This means that the curve and the envelope have the same tangent at P. In other words, they touch each other at P.

If $\partial F/\partial x$ and $\partial F/\partial y$ are both zero, the value of dy/dx cannot be found from equation (5) or (6), and the above argument would break down. So the proposition might not be true for such points. We have seen before (§ 10·25) that if $\partial F/\partial x = \partial F/\partial y = 0$ at some point, then there is a singular point there.

12·21. Equivalence of the two definitions. The proposition of the last article enables us to infer at once that in general the two definitions of an envelope given in §§ 12·11 and 12·12 would give us the same curve, with the exception that the second definition might in certain cases give us a curve the whole or a part of which is not an envelope in the sense of the first definition.

For example, if the curve $F(x, y, a) = 0$ is the curve C_1 (which has a cusp at P), and C_2 is the curve $F(x, y, a+h) = 0$, it is evident that as $h \to 0$, i.e., as $C_2 \to C_1$, P_1 tends to P. Hence the result of eliminating a between $F = 0$ and $\partial F/\partial a = 0$ will be, or will at least include, the locus of the cusps. But from the figure it is evident that the locus of the cusps will not touch C_1, or C_2, or the other members of the family.



There are other loci (besides the locus of the cusps) which are sometimes obtained in the process of finding the envelope by eliminating a between $F = 0$ and $\partial F/\partial a = 0$. The consideration of the subject in detail, however, is beyond the scope of the present volume.

12·3. Further Examples.

If the equation to a family of curves is not given, but the law is given in accordance with which any member of the family can be obtained, the equation to the family must first be found in a suitable form.

Ex. 1. Find the envelope of the circles drawn upon the radii vectores of the ellipse $x^2/a^2 + y^2/b^2 = 1$ as diameter.

Any point on the ellipse is $(a \cos\theta, b \sin\theta)$. So the equation of the circle on the radius vector to this point as diameter is

$$x^2 + y^2 - ax\cos\theta - by\sin\theta = 0.$$

Differentiating w. r. t. θ,

$$ax \sin\theta - by \cos\theta = 0.$$

whence

$\tan\theta = by/ax$, $\sin\theta = by/(a^2x^2 + b^2y^2)^{1/2}$, $\cos\theta = ax/(a^2x^2 + b^2y^2)^{1/2}$.

Hence the envelope is

$$x^2 + y^2 - \frac{a^2x^2}{(a^2x^2 + b^2y^2)^{1/2}} - \frac{b^2y^2}{(a^2x^2 + b^2y^2)^{1/2}} = 0,$$

or $$x^2 + y^2 - (a^2x^2 + b^2y^2)^{1/2} = 0,$$

i.e., $$(x^2 + y^2)^2 = a^2x^2 + b^2y^2.$$

Ex. 2. Find the envelope of the circles described on the radii vectores of the curve $r^n = a^n \cos n\theta$ as diameter.

Here it would be convenient to take the polar equation of the circle. If (R, Θ) be the polar coordinates of any point of the circle, its equation is

$$R = r\cos(\Theta - \theta),$$

or $$R = a\cos^{1/n} n\theta \cos(\Theta - \theta). \quad . \quad . \quad . \quad . \quad . \quad (1)$$

Differentiating logarithmically w. r. t. θ, which is the parameter,

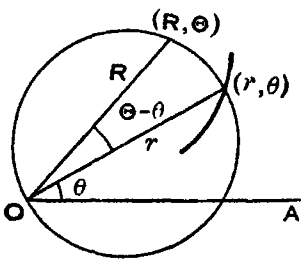
$$0 = -(1/n) \,.\, n \,.\, \tan n\theta + \tan(\Theta - \theta).$$

Hence $$\tan(\Theta - \theta) = \tan n\theta.$$

Therefore $$\Theta - \theta = n\theta + m\pi,$$

or $$\theta = \Theta/(n+1) - m\pi/(n+1).$$

By substituting this value of θ in (1) we get the envelope.



Ex. 3. Find the envelope of the straight lines drawn at right angles to the radii vectores of the spiral $r = ae^{\theta \cot \alpha}$ through their extremities.

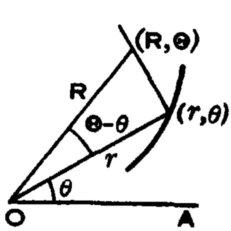
The equation to the line through (r, θ), at right angles to the radius vector, is

$$R\cos(\Theta - \theta) = r.$$

i.e., $$R\cos(\Theta - \theta) = ae^{\theta\cot\alpha}, \quad . \quad . \quad (1)$$

Differentiating logarithmically w. r. t. θ, the parameter, we get

$$\tan(\Theta - \theta) = \cot\alpha = \tan(\tfrac{1}{2}\pi - \alpha).$$



Therefore $\Theta - \theta = \frac{1}{2}\pi - \alpha + m\pi$, or $\theta = \Theta + \alpha - \frac{1}{2}(2m + 1)\pi$.
Substituting this in (1), the required envelope is

$$R \cos\{\alpha - \tfrac{1}{2}(2m+1)\pi\} = ae^{\{\Theta + \alpha - \frac{1}{2}(2m+1)\pi\}\cot\alpha},$$

which is of the form $R = A\,e^{\Theta \cot \alpha}$.

Hence the envelope is another spiral of the same type.

12·31. Two parameters connected by a relation. When the equation of a family of curves contains two parameters connected by a relation, and the elimination of one of them from the equation of the family makes the subsequent process of finding the envelope tedious, we can proceed as in the following example:

Ex. Find the envelope of the straight lines

$$\frac{x}{a} + \frac{y}{b} = 1, \qquad \dots\dots\dots \quad (1)$$

where the parameters a and b are related by the equation

$$a^n + b^n = c^n, \qquad \dots\dots\dots \quad (2)$$

c being a constant.

Let us regard (for the sake of symmetry) a and b as functions of t. Differentiating (1) with respect to t (which is now the only parameter), we have

$$\frac{x}{a^2}\frac{da}{dt} + \frac{y}{b^2}\frac{db}{dt} = 0.$$

We have first to simplify this. So differentiating (2) with respect to t, we have

$$a^{n-1}\frac{da}{dt} + b^{n-1}\frac{db}{dt} = 0.$$

Equating the values of $\dfrac{da}{dt}\Big/\dfrac{db}{dt}$ obtained from these equations, we have

$$\frac{\dfrac{x}{a^2}}{a^{n-1}} = \frac{\dfrac{y}{b^2}}{b^{n-1}}. \qquad \dots\dots\dots \quad (3)$$

We have thus to eliminate a and b between (1), (2) and (3). Now (3) gives

$$\frac{\dfrac{x}{a}}{a^n} = \frac{\dfrac{y}{b}}{b^n} = \frac{\dfrac{x}{a} + \dfrac{y}{b}}{a^n + b^n} = \frac{1}{c^n},$$

whence $$a^{n+1} = c^n x, \quad b^{n+1} = c^n y.$$

Substituting in (2), we have

$$(c^n x)^{n/(n+1)} + (c^n y)^{n/(n+1)} = c^n,$$

or $$x^{n/(n+1)} + y^{n/(n+1)} = c^{n(n+1)},$$

which is the equation of the required envelope.

EXAMPLES

1. Find the envelopes of circles described on the radii vectores of the following curves as diameters:

(i) $y^2 = 4ax$. (ii) $l/r = 1 + e\cos\theta$.

(iii) $r^3 = a^3 \cos 3\theta$. (iv) $r\cos^n(\theta/n) = a$.

2. Find the envelopes of the straight lines, drawn through the extremities of, and at right angles to, the radii vectores of the following curves:

(i) $r\cos(\theta + a) = p$. (ii) $r = a(1 + \cos\theta)$. [*Aligarh*, 1930]

(iii) $r^n \cos n\theta = a^n$. (iv) $r^n = a^n \cos n\theta$.

3. Find the envelope of the straight line $x/a + y/b = 1$ when $a^m b^n = c^{m+n}$, where c is a constant.

4. Prove that the envelope of ellipses having the axes of coordinates as principal axes and the sum of their semi-axes constant and equal to c, is the astroid

$$x^{2/3} + y^{2/3} = c^{2/3}.$$ [*Benares*, 1934]

5. Find the envelope of a system of concentric and coaxial ellipses of constant area. [*Patna*, 1933]

12·4. Evolute. *The locus of the centre of curvature for a curve is called its evolute.*

Ex. Find the evolute of the parabola $y^2 = 4ax$.

By § 9·2, $$\alpha = 2a + 3x, \quad \beta = -2a^{-1/2} x^{3/2}.$$

Eliminating x between these equations, we have

$$\beta^2 = 4a^{-1} x^3 = 4a^{-1}(\alpha - 2a)^3/27,$$

i.e., $$27a\beta^2 = 4(\alpha - 2a)^3.$$

Replacing α and β by x and y respectively, we have as the equation of the evolute

$$27ay^2 = 4(x - 2a)^3.$$

12·41. Envelope of the normals. As the centre of curvature of a curve for a given point A on it is the limiting position of the intersection of the normal at A with the normal at any other point B, as $B \to A$, and the evolute is the locus of the centre of curvature, it follows by § 12·12 that *the evolute of a curve is the envelope of the normals of that curve.*

Therefore (by § 12·2) *the normals of a curve touch the evolute.**

We have thus an alternative method of finding the evolute.

Ex. Find the evolute of the parabola $y^2 = 4ax$.

Any normal to the parabola is

$$y = mx - 2am - am^3,$$

where a is a constant and m the parameter.

Differentiating w. r. t. m, we have

$$0 = x - 2a - 3am^2,$$

i.e.,
$$m = \left(\frac{x - 2a}{3a}\right)^{1/2}.$$

Substituting this in the equation of the normal, we get the evolute

$$y = \left(\frac{x-2a}{3a}\right)^{1/2} \left(x - 2a - a \,.\, \frac{x-2a}{3a}\right),$$

or
$$27ay^2 = 4(x - 2a)^3,$$

as before

EXAMPLES

1. Find the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$.
[*Allahabad*, 1933]

2. Prove that the evolute of the hyperbola $2xy = a^2$ is
$$(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}.$$

3. Prove that the evolute of the tractrix
$$x = a(\cos t + \log \tan \tfrac{1}{2}t), \quad y = a \sin t,$$
is the catenary
$$y = a \cosh (x/a).$$

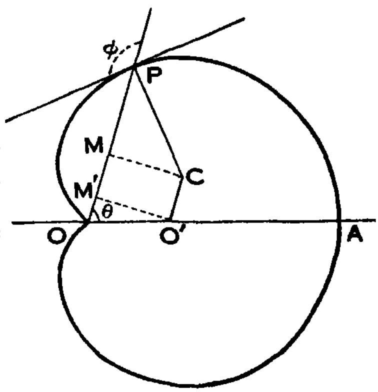
*The exceptional cases mentioned in § 12·21 evidently cannot arise in the case of envelopes of *straight lines*.

4. Prove that the evolute of the cardioid $r = a(1 + \cos\theta)$ is the cardioid

$$r = \tfrac{1}{3}a(1 - \cos\theta),$$

the pole in the latter equation being at the point $(\tfrac{2}{3}a, 0)$.

[There is no standard method of finding the evolute of a curve given by its polar equation. This question can be done as follows:



Take any point P (r, θ) on the given cardioid. Let O' be the point $r = \tfrac{2}{3}a$, $\theta = 0$. Let C be the centre of curvature at P, and let $O'C = \mathrm{R}$, $\angle CO'A = \Theta$. Let CM and $O'M'$ be perpendiculars dropped on OP. Then

$PC = \tfrac{4}{3}a\cos\tfrac{1}{2}\theta$, see Ex. 12, p. 135.

Also $r\,d\theta/dr = -\cot\tfrac{1}{2}\theta$;

so $\phi = \tfrac{1}{2}\pi + \tfrac{1}{2}\theta$, and $\angle CPM = \tfrac{1}{2}\theta$.

Therefore $CM = PC\sin\tfrac{1}{2}\theta = \tfrac{4}{3}a\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta$,

and $$O'M' = OO'\sin\theta = \tfrac{2}{3}a\sin\theta = CM.$$

Therefore $O'C$ is parallel to OP, i.e., $\Theta = \theta$. . . . (1)

Also $\mathrm{R} = r - OM' - MP = r - \tfrac{2}{3}a\cos\theta - \tfrac{4}{3}a\cos^2\tfrac{1}{2}\theta$ (2)

$$= \tfrac{1}{3}a(1 - \cos\theta).$$

Substituting from (1) in (2), we have

$$\mathrm{R} = \tfrac{1}{3}a(1 - \cos\Theta)$$

as the equation of the locus of C.]

5. Prove that the evolute of an equiangular spiral is another equiangular spiral with the same pole. [*Mysore*, 1936]

[Start with the pedal equation $p = r\sin\alpha$, where α is a constant, and prove geometrically that if any point on the evolute is (P, R), then $P = \mathrm{R}\sin\alpha$, using the fact that $\phi = \alpha$.]

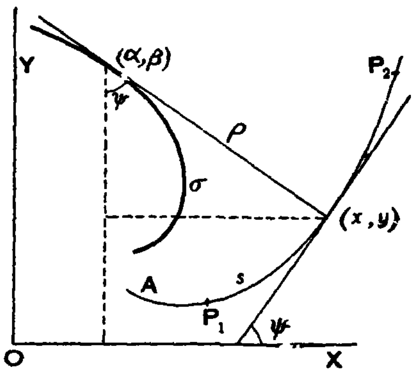
6. Show that the evolute of the curve whose pedal equation is

$$r^2 - a^2 = mp^2$$

is the curve whose pedal equation is

$$r^2 - (1 - m)a^2 = mp^2.$$

12·42. Length of arc of an evolute. *The difference between the radii of curvature at any two points of a curve is equal to the length of the arc of the evolute between the two corresponding points.*



A suitable sign being given to ϱ, it is evident from the figure that, if (α, β) is the centre of curvature corresponding to (x, y) on the curve,

$$\left.\begin{aligned} \alpha &= x - \varrho \sin\psi, \\ \beta &= y + \varrho \cos\psi. \end{aligned}\right\} \qquad (1)$$

Let s be the length of the arc of the given curve measured from some fixed point A on the curve up to (x, y), and σ the length of the arc of the evolute measured from some fixed point on it up to (α, β).

Remembering that

$$dx/ds = \cos\psi, \quad dy/ds = \sin\psi, \quad \text{and } d\psi/ds = 1/\varrho,$$

we have, by differentiating (1) and simplifying,

$$\frac{d\alpha}{ds} = -\frac{d\varrho}{ds}\sin\psi, \quad \frac{d\beta}{ds} = \frac{d\varrho}{ds}\cos\psi.$$

Squaring, adding, and taking the square root,

$$\frac{d\varrho}{ds} = \left\{\left(\frac{d\alpha}{ds}\right)^2 + \left(\frac{d\beta}{ds}\right)^2\right\}^{\frac{1}{2}}$$

$$= \left\{1 + \left(\frac{d\beta}{d\alpha}\right)^2\right\}^{\frac{1}{2}} \cdot \frac{d\alpha}{ds}$$

$$= \frac{d\sigma}{d\alpha} \cdot \frac{d\alpha}{ds} = \frac{d\sigma}{ds}.$$

Hence $$\frac{d\varrho}{d\sigma} = 1.$$

Therefore $\sigma = \varrho + c$, where c is a constant.

Hence $\sigma_2 - \sigma_1 = \varrho_2 - \varrho_1$, where ϱ_1 and ϱ_2 are the values of ϱ for any two points ϱ_1 and ϱ_2 on the curve, and σ_1 and σ_2 are the corresponding values of σ.

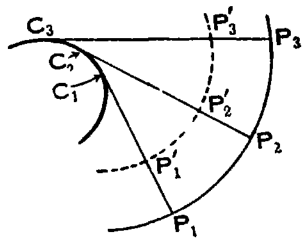
This proves the proposition.

Examples

1. Show that the whole length of the evolute of the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $4(a^2/b - b^2/a)$.

2. Find the evolute of the parabola and show that the length of the arc of the evolute from the cusp to the point at which the evolute meets the parabola is $2a(3\sqrt{3} - 1)$, where $4a$ is the latus rectum of the parabola. [*Allahabad*, 1927]

12·43. Involutes. If one curve is the evolute of another, then the latter is called an *involute* of the former.



Thus if the curve $C_1 C_2 C_3$ is the evolute of the curve $P_1 P_2 P_3$, then $P_1 P_2 P_3$ is an involute of $C_1 C_2 C_3$.

If C_1 and C_2 are the centres of curvature of the curve $P_1 P_2 P_3$ at P_1 and P_2 respectively, then by the last article

$$C_1 P_1 + \text{arc } C_2 C_1 = C_2 P_2$$

Hence, if a thread were wrapped round the curve $C_3 C_2 C_1$ and were prevented from slipping, it is evident that when the thread is unwrapped, being kept taut all the time, the point on the thread which was at P_1 to begin with will describe the curve $P_1 P_2 P_3$.

This explains why the curve $C_1 C_2 C_3$ is called the *evolute* of the curve $P_1 P_2 P_3$.

Obviously any point on the thread will describe an involute of the curve $C_1 C_2 C_3$. Thus *every curve has an infinite number of involutes.*

If the curves $P_1 P_2 P_3$ and $P_1' P_2' P_3'$ are both involutes of the same curve, then they are called *parallel curves*, because the distance between them measured along their common normal is constant.

Examples on Chapter XII

1. A straight line cuts off from the axes of coordinates intercepts OA and OB, such that $n.OA + OB = c$. Show that its envelope is

$$(y - nx - c)^2 = 4ncx.$$

2. Find the envelope of the family of curves $(a^2/x) \cos\theta - (b^2/y) \sin\theta = c$ for different values of θ. [*Patna*, 1931]

3. Find the envelope of the circles which pass through the origin and whose centres lie on

(i) $x^2/a^2 + y^2/b^2 = 1$.

(ii) $r^2 \cos 2\theta = a^2$.

4. A circle moves with its centre on the parabola $y^2 = 4ax$ and always passes through the vertex of the parabola. Show that the envelope of the circle is the curve $x^3 + y^2(x + 2a) = 0$. [*Agra*, 1928]

5. Show that the envelope of the family of circles whose diameters are double ordinates of the parabola $y^2 = 4ax$ is the parabola

$$y^2 = 4a(a + x).$$

6. Find the envelope of $x \cos^n\theta + y \sin^n \theta = a$.

[*Allahabad*, 1927]

7. Show that the envelope of circles described on the central radii of a rectangular hyperbola is a lemniscate

$$r^2 = a^2 \cos 2\theta.$$ [*Madras*, 1937]

8. Find the envelope of the curves

$$\frac{x^m}{a^m} + \frac{y^m}{b^m} = 1,$$

where the parameters a and b are connected by the relation

$$a^p + b^p = c^p.$$

9. Show that the envelope of the family of parabolas

$$(x/a)^{1/2} + (y/b)^{1/2} = 1,$$

under the condition $ab = c^2$, is a hyperbola having its asymptotes coinciding with the axes.

10. Projectiles are fired from a gun with a constant initial velocity v_0. Supposing the gun can be given any elevation and is kept always in the same vertical plane, what is the envelope of all possible trajectories, assuming their equation to be

$$y = x \tan \alpha - \frac{gx^2}{2v_0^2 \cos^2 \alpha}?$$

11. A straight line of given length slides with its extremities on two fixed straight lines at right angles. Find the envelope of the circle drawn on the sliding line as diameter.

12. Obtain the envelope of the family of curves given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{k^2 - \alpha^2} = 1,$$

where α is the parameter. [*Allahabad*, 1930]

13. From any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$, perpendiculars are drawn to the axes, and the feet of these perpendiculars are joined. Show that the straight line thus formed always touches the curve

$$(x/a)^{2/3} + (y/b)^{2/3} = 1.$$ [*Agra*, 1932]

14. Show that the envelope of the polars of points on the ellipse $x^2/h^2 + y^2/k^2 = 1$ with respect to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is

$$\frac{h^2x^2}{a^4} + \frac{k^2y^2}{b^4} = 1.$$

15. Show that the envelope of the straight lines

$$y \cos\theta - x \sin\theta = a - a \sin\theta \log \tan (\tfrac{1}{2}\theta + \tfrac{1}{4}\pi),$$

where θ is the parameter, is the catenary

$$y = a \cosh (x/a).$$

16. Find the envelope of the ellipse

$$x = a \sin (\theta - \alpha), \; y = b \cos\theta,$$

where α is the parameter.

17. Find the envelope of a family of parabolas, of given latus rectum and parallel axes, when the locus of their foci is a given straight line $y = px + q$. [*Punjab*, 1937]

18. Show that the envelope of the straight line joining the extremities of a pair of conjugate diameters of an ellipse is a similar ellipse.

19. Prove that the evolute of the ellipse

$$b^2x^2 + a^2y^2 = a^2 b^2$$

is the envelope of the family of ellipses given by

$$a^2x^2 \sec^4 \alpha + b^2y^2 \operatorname{cosec}^4 \alpha = (a^2 - b^2)^2,$$

α being the variable parameter. [*Benares*, 1928]

20. Show that the radius of curvature of the envelope of the line $x \cos \alpha + y \sin \alpha = f(\alpha)$ is

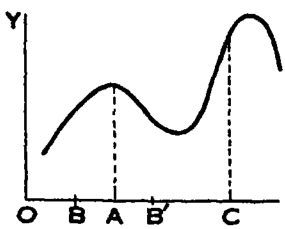
$$f(\alpha) + f''(\alpha).$$ [*Agra*, 1935]

CHAPTER XIII

MAXIMA AND MINIMA

13·1. Definitions. A function $f(x)$ is said to have a *maximum* for a value a of x if $f(a)$ is greater than the value of the function for every other value of x in a small neighbourhood of a, i.e., if an ε can be found such that $f(x)-f(a)$ is negative for all values of x for which

$$0 < |x-a| < \varepsilon.$$



Thus, if the accompanying figure shows a graph of the function $f(x)$, and $OA = a$, we say that $f(x)$ has a maximum for $x = a$, because $f(a)$ is greater than the values of $f(x)$ for every value of x between B and B' (except, of course, $x = a$ itself, where the inequality reduces to an equality). It will be noticed that $f(x)$ is said to be a maximum at $x = a$, even though the value of $f(x)$ at $x = a$ is less than the value of $f(x)$ at C. What the definition requires is that $f(x)$ at $x = a$ should be greater than all other vlues of $f(x)$ in some small neighbourhood. Thus a maximum value of $f(x)$ is not necessarily the greatest value of $f(x)$. In fact, a curve might have several maxima (and minima).

A function $f(x)$ is said to have a *minimum* for a value a of x if $f(a)$ is less than the value of the function for every other value of x in a small neighbourhood of a, i.e., if an ε can be found such that $f(x)-f(a)$ is positive for all values of x for which $0 < |x-a| < \varepsilon$.

13·12. Conditions for maxima and minima. Let $f(x)$ be a given function of x, and suppose $f(x)$ is expansible in the neighbourhood of $x = a$ by Taylor's theorem.

Then, if $0 < \theta < 1$,

$$f(x) = f(a) + (x - a)f'(a) + (1/2!)(x - a)^2 f''(a) + \dots + \{1/(n-1)!\}(x-a)^{n-1} f^{(n-1)}(a) + (1/n!)\ (x - a)^n f^n\{a + \theta(x - a)\}.$$

That is,

$$f(x) - f(a) = (x - a)[f'(a) + (x - a)\{(1/2!)f''(a) + (1/3!)(x - a)f'''(a) + \dots + (1/n!)(x - a)^{n-2} f^{(n)}(a + \theta\,\overline{x - a})\}]. \quad (1)$$

I. If $f(x)$ is a maximum at $x = a$, then by definition, $f(x) - f(a)$, and therefore also the right-hand side of (1), must be negative for sufficiently small values of $|x - a|$, whether $x - a$ be positive or negative.

But the series which occurs inside the crooked brackets in the right-hand side of (1) is a finite expression which does not tend to infinity as $x \to a$. Hence the product of this expression and $(x - a)$ can be made as small as we please by making $|x - a|$ sufficiently small. If, therefore, $f'(a)$ is not equal to zero, we can make the product numerically less than $f'(a)$. So, for sufficiently small values of $|x - a|$, the sign of the expression within the square brackets in (1) will be the same as that of $f'(a)$.

It follows, therefore, that if $f'(a) \neq 0$, $f(x) - f(a)$ will have one sign when $x - a$ is positive and the opposite if $x - a$ is negative. Hence $f(x)$ cannot have a maximum at $x = a$ if $f'(a) \neq 0$.

Similarly, since for a minimum at $x = a$, $f(x) - f(a)$ must (for sufficiently small values of $|x - a|$) be positive, whether $x - a$ is positive or negative, it follows that $f(x)$ cannot have a *minimum* also at $x = a$ if $f'(a) \neq 0$.

Hence a *necessary condition that $f(x)$ should have a maximum or a minimum at $x = a$ is that*

$$\mathbf{f'(a) = 0.}$$

II. If $f'(a) = 0$, then by (1)

$$f(x) - f(a) = (x - a)^2\{(1/2!)f''(a) + (1/3!)(x - a)f'''(a) + \dots + (1/n!)(x - a)^{n-2} f^{(n)}(a + \theta\,\overline{x - a})\}.$$

Now the sign of the expression between the crooked brackets is, for sufficiently small values of $|x - a|$, the same as that of $f''(a)$. Also the sign of $(x-a)^2$ is always positive $(x \neq a)$. Hence $f(x) - f(a)$ will be negative for sufficiently small values of $|x - a|$ and irrespective of whether $x > a$ or $x < a$, if $f''(a)$ is negative.

Thus *there is a* **maximum** *of* $f(x)$ *at* $x = a$ *if* $\mathbf{f'(a) = 0}$ *and* **f″(a) is negative.**

Similarly, *there is a* **minimum** *of* $f(x)$ *at* $x = a$ *if* $\mathbf{f'(a) = 0}$ *and* **f″(a) is positive.**

13·13. **Generalisation.** The above argument shows also that if $f'(a) = f''(a) = f'''(a) = \ldots = f^{(n-1)}(a) = 0$, and $f^{(n)}(a) \neq 0$, then for a maximum or minimum, we must have n even. Also, for a maximum $f^{(n)}(a)$ must be negative, and for a minimum $f^{(n)}(a)$ must be positive.

NOTE. For the validity of this proposition it is only necessary that $f(x)$ and its first n differential coefficients be continuous at $x = a$. The higher differential coefficients of $f(x)$ may even be non-existent: the propositions will still be true. This becomes evident from (1) of the last article when we notice that $f(x) - f(a)$ now reduces to

$$(1/n!)(x-a)^n f^{(n)}\{a + \theta(x-a)\},$$

and the sign of this must, on account of the continuity of $f^n(x)$ at $x = a$, be the same as that of $(x-a)^n f^{(n)}(a)$ for sufficiently small values of $x - a$.

13·14. **Working Rule.** The proposition of § 13·12 gives us the following rule for determining maxima and minima.

Find $f'(x)$ and equate it to zero. Solve this equation for x. Let the roots be $a_1, a_2, \ldots$ Find $f''(x)$ and substitute in it by turns $a_1, a_2, \ldots$.

If $f''(a_1)$ is negative, we have a maximum at $x = a_1$. If $f''(a_1)$ is positive, we have a minimum at $x = a_1$.

If $f''(a_1) = 0$, we must find $f'''(x)$ and substitute in it a_1. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at a_1. But if $f'''(a_1)$ is also zero, we must find $f^{iv}(x)$ and substitute in it a_1. If $f^{iv}(a_1)$ is negative, we have a maximum at $x = a_1$; if it is positive, a minimum. If it is zero, we must find $f^{v}(x)$, and so on.

Ex. 1. Find the maximum value of $(x-1)(x-2)(x-3)$.

Let $f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6$.
Then, for a maximum or a minimum, $f'(x) = 0$, i.e.,

$$3x^2 - 12x + 11 = 0,$$

whence
$$x = \frac{6 \pm \sqrt{(36-33)}}{3} = 2 \pm 1/\sqrt{3}.$$

Also $f''(x) = 6x - 12$.

We see that $f''(2 + 1/\sqrt{3})$ is positive and $f''(2 - 1/\sqrt{3})$ is negative.

Hence $(x-1)(x-2)(x-3)$ has a maximum at $x = 2 - 1/\sqrt{3}$, and a minimum at $x = 2 + 1/\sqrt{3}$.

The maximum value of $(x-1)(x-2)(x-3)$ is, therefore,

$$(1 - 1/\sqrt{3})(-1/\sqrt{3})(-1 - 1/\sqrt{3}),$$

i.e.,
$$(1 - \tfrac{1}{3})(1/\sqrt{3}), \text{ or } 2/3\sqrt{3}$$

13·15. Properties of Maxima and Minima. If $f(x)$ is a continuous function and so has as its graph an unbroken curve, a figure will at once show that:

(1) *At least one maximum or one minimum must lie between two equal values of* $f(x)$.

(2) *Maxima and minima occur alternately.*

(3) *The sign of* dy/dx *changes from* + *to* − *as* x *passes (while increasing) through the value which makes* y *a maximum.* The reverse is the case for a minimum. See § 4 31.

These considerations often enable us to decide, without finding d^2y/dx^2, whether there is a maximum or a minimum at a particular root of $f'(x) = 0$.

Ex. A rectangular sheet of metal has four equal square portions removed at the corners, and the sides are then turned up so as to form an open rectangular box. Show that when the volume contained in the box is a maximum, the depth will be

$$\tfrac{1}{6}\{(a+b) - (a^2 - ab + b^2)^{1/2}\},$$

where a, b are the sides of the original rectangle. [*Allahabad*, 1931]

Let x be the length of the side of each of the square portions removed. Then the volume V of the box formed

$$= (a - 2x)(b - 2x)x \quad . \quad . \quad . \quad . \quad (1)$$
$$= 4x^3 - 2x^2(a+b) + abx.$$

Therefore, for a maximum or a minimum V, we must have $dV/dx = 0$,

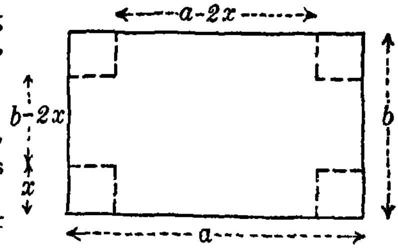
i.e.,
$$12x^2 - 4x(a+b) + ab = 0,$$

or
$$x = \frac{2(a+b) \pm \sqrt{\{4(a+b)^2 - 12ab\}}}{12}$$
$$= \tfrac{1}{6}\{(a+b) \pm (a^2 - ab + b^2)^{1/2}\}.$$

As $a^2 - ab + b^2 = (a - b)^2 + ab$, and is therefore positive, the values we have found are real. But the value of x obtained by taking the positive sign before the radical, viz., $\frac{1}{6}\{(a + b) + (a^2 - ab + b^2)^{1/2}\}$ is (if $a > b$) greater than

$\frac{1}{6}\{(b + b) + (ab - ab + b^2)^{1/2}\}$,

i.e., is greater than $\frac{1}{2}b$, which is impossible (see figure).



Hence we have a maximum or minimum volume only when

$$x = \tfrac{1}{6}\{(a + b) - (a^2 - ab + b^2)^{1/2}\}. \quad . \quad . \quad . \quad . \qquad (2)$$

But $V = 0$ when $x = 0$, and also when $x = \frac{1}{2}b$, and for intermediate values V is positive. Hence the value (2) of x must give a maximum V.

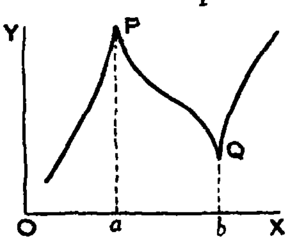
13·16. Stationary values. dy/dx is the rate at which y increases (§ 4·3).

So, if dy/dx is zero at $x = a$, we say that y is *stationary* at $x = a$.

Similarly d^2y/dx^2 is the rate at which dy/dx increases. So, if d^2y/dx^2 is zero at $x = a$, we say that *the tangent to the curve at* $x = a$ *is stationary*.

It is obvious from the above that $f(x)$ is stationary when it is a maximum or a minimum. At a point of inflexion the tangent is stationary, but we have neither a maximum nor a minimum there, even if dy/dx is zero there, i.e., even if the tangent there is parallel to the x-axis.

13·17. Points where the tangent is vertical. At a point like P or Q where we have a cusp with a vertical tangent, dy/dx is discontinuous, for at P its limit on the right is $-\infty$, and its limit on the left is $+\infty$. So d^2y/dx^2 does not exist at P. Similarly it does not exist at Q. Hence such points will not be obtained by the preceding methods. Fortunately, however, such cases do not occur in practical problems.



EXAMPLES

1. Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maximum when $x = 1$, a minimum when $x = 3$, and neither when $x = 0$.

2. Show that $x^3 - 3x^2 + 3x + 7$ has neither a maximum nor a minimum at $x = 1$.

3. Show that $x^3 - 3x^2 + 6x + 7$ has no maxima or minima.

4. Find the stationary points of the function $x^5 - 5x^4 + 5x^3 + 1$ and examine for which of them the function is a maximum or a minimum. [*Aligarh*, 1934]

5. Find the maxima and minima, if any, of

$$x^4/(x-1)(x-3)^3.$$ [*Dacca*, 1937]

6. Prove that $x/(1 + x \tan x)$ is a maximum when $x = \cos x$.

7. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$. [*Patna*, 1934]

8. Show that $\sin x.(1 + \cos x)$ is a maximum when $x = \frac{1}{3}\pi$.

9. If $dy/dx = (x-a)^{2n}(x-b)^{2p+1}$, where n and p are positive integers, show that $x = a$ gives neither a maximum nor a minimum value of y, but $x = b$ gives a minimum. [*Lucknow*, 1932]

10. If $xy(y-x) = 2a^3$, show that y has a minimum value when $x = a$.

11. Show that $\sin^p\theta \cos^q\theta$ attains a maximum when

$$\theta = \tan^{-1}\sqrt{(p/q)}.$$ [*Benares*, 1926]

12. A person being in a boat a miles from the nearest point of the beach, wishes to reach as quickly as possible a point b miles from that point along the shore. The ratio of his rate of walking to his rate of rowing is $\sec \alpha$. Prove that he should land at a distance $b - a \cot \alpha$ from the place to be reached. [*Madras*, 1936]

13. Prove that the area of the triangle formed by the tangent to a given curve and the axes of coordinates is a maximum or a minimum when the point of contact is the middle point of the hypotenuse. [*Benares*, 1931]

14. Show that the altitude of the greatest equilateral triangle that can be circumscribed about a given triangle is

$$\{a^2 + b^2 - 2ab \cos(\tfrac{1}{3}\pi + C)\}^{1/2}.$$

15. The velocity of waves of wave-length λ on deep water is proportional to $\sqrt{(\lambda/a + a/\lambda)}$, where a is a certain linear magnitude: prove that the velocity is a minimum when $\lambda = a$.

16. If the sum of the lengths of the hypotenuse and another side of a right angled triangle is given, show that the area of the triangle is a maximum when the angle between these sides is $\frac{1}{3}\pi$. [*Agra*, 1930]

17. A perpendicular is let fall from the centre on a tangent to an ellipse. Show that the maximum value of the intercept between the point of contact and the foot of the perpendicular is $a - b$, where a and b are the semi-axes of the ellipse. [*Nagpur*, 1931]

18. Prove that the minimum radius vector of the curve $a^2/x^2 + b^2/y^2 = 1$ is of length $a + b$.

19. Find the maxima and minima of the radii vectores of the curve

$$\frac{c^4}{r^2} = \frac{a^2}{\sin^2\theta} + \frac{b^2}{\cos^2\theta}.$$

20. Show that the maximum and minimum values of $x^2 + y^2$, where $ax^2 + 2hxy + by^2 = 1$, are given by the roots of the quadratic

$$(a - 1/r^2)(b - 1/r^2) = h^2.$$

Hence find the area of the conic denoted by the first equation.

[*Allahabad*, 1937]

21. The three sides of a trapezium are equal, each being 6 inches long ; find the area of the trapezium when it is a maximum.

[*Delhi*, 1935]

22. Find the maximum value of $(\log x)/x$ in $0 < x < \infty$.

[*Bombay*, 1937]

23. Prove that the maximum and minimum values of the function $y = (ax^2 + 2bx + c)/(Ax^2 + 2Bx + C)$ are those values of y for which

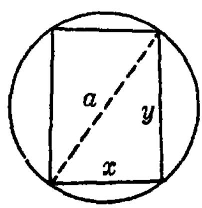
$$(ax^2 + 2bx + c) - y(Ax^2 + 2Bx + C)$$

is a perfect square. [*Mysore*, 1937]

13·18. Two variables connected by a relation.

If the function whose maxima and minima are to be found is expressed as a function of two variables connected a given relation, one of the variables should first be eliminated.

Ex. Assuming that the stiffness of a beam of rectangular cross-section varies as the breadth and as the cube of the depth, what must be the breadth of the stiffest beam that can be cut from a log of diameter a?



Let the breadth be x and the depth y, and let S denote the stiffness.

Then $S = kxy^3$, where k is a constant.

But $x^2 + y^2 = a^2$.

Hence $S = kx(a^2 - x^2)^{3/2}$.

For a maximum or a minimum S, we must have $dS/dx = 0$,

i.e., $$(a^2 - x^2)^{3/2} + x \cdot \tfrac{3}{2}(a^2 - x^2)^{1/2}(-2x) = 0;$$

or $$a^2 - x^2 - 3x^2 = 0, \text{ i.e., } x = \pm \tfrac{1}{2}a.$$

Hence for the stiffest beam the breadth must be equal to half the diameter of the log.

EXAMPLES

1. Find the maximum and minimum values of

 (i) $ax + by$, when $xy = c^2$;

 (ii) $\sin^2\theta + \sin^2\phi$, when $\theta + \phi = \alpha$.

2. A figure consists of a semicircle with a rectangle on its diameter. Given that the perimeter of the figure is 20 feet, find its dimensions in order that its area may be a maximum. [*Allahabad*, 1926]

3. An open rectangular tank, with a square base and vertical sides, is to be constructed of sheet metal to hold a given quantity of water. Show that the cost of material will be least when the depth is half the width.

4. The strength of a beam varies as the product of its breadth and the square of its depth. Find the dimensions of the strongest beam which can be cut from a circular log of radius a. [*Nagpur*, 1925]

5. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

6. A lane runs at right angles out of a road 18 feet wide. Find to one decimal place how many feet wide the lane must be if it is just possible to carry a pole 45 feet long from the road into the lane, keeping it horizontal. [*Math. Tripos*, 1934]

7. A ray of light travels in a plane perpendicular to the edge of a prism of angle i. If the angle of incidence is ϕ and the angle of emergence ϕ', show that the deviation $\phi + \phi' - i$ is a minimum when $\phi = \phi'$. [*Punjab*, 1929]

[By the laws of Physics, $\sin\phi/\sin r_1 = \sin\phi'/\sin r_2 = \mu$, where μ is a constant, and $r_1 + r_2 = i$.]

EXAMPLES ON CHAPTER XIII

1. Show that the maximum rectangle inscribable in a circle is a square. [*Calcutta*, 1936]

2. Determine for what values of x the function $12x^5 - 5x^4 + 40x^3 + 6$ acquires maximum and minimum values. [*All.*, 1929]

3. Investigate the maxima and minima of $\sin nx \sin^n x$, where n is a positive integer.

4. Find the largest and the smallest values of the polynomial $x^3 - 18x^2 + 96x$ in the interval $(0, 9)$. [*Math. Tripos*, 1931]

5. Divide a into two parts such that the product of the pth power of one and the qth power of the other is as great as possible.

6. A gas-holder is a cylindrical vessel closed at the top and open at the bottom (which dips into water). What should be the ratio of

[illegible]

the height to the diameter in order that its construction may require the least amount of material?

7. Assuming that the petrol burnt in driving a motor boat varies as the cube of its velocity, show that the most economical speed when going against a current of c miles per hour is $\frac{3}{2}c$ miles per hour.

8. The lower corner of a leaf in a book is folded over so as just to reach the inner edge of the page. Show that the fraction of the width folded over when the area of the folded part is a minimum is $\frac{2}{3}$.

9. The theory of probabilities shows that if $x_1, x_2, \ldots x_n$ are the measures of an unknown magnitude x, so that the errors are $x - x_1, x - x_2, \ldots x - x_n$, the most probable value of x is that which makes the sum of the squares of the errors a minimum. Show that, on this theory, the most probable value of the unknown is the arithmetic mean of the measures.

10. An electric light is placed directly over the centre of a circular plot of lawn 100 feet in diameter. Assuming that the intensity of light varies directly as the sine of the angle at which it strikes an illuminated surface, and inversely as the square of its distance from the surface, how high should the light be hung in order that the intensity may be as great as possible at the circumference of the plot? [*Aligarh*, 1930]

11. Find the area of the greatest isosceles triangle that can be inscribed in a given ellipse, the triangle having its vertex coincident with one extremity of the major axis. [*Agra*, 1934]

12. A straight line is drawn through the point (h, k), cutting off intercepts a and b from the axes of x and y respectively. Show that if the acute angle between the straight line and the axis of x is θ, $a^n + b^n$ is a minimum when

$$\tan \theta = (k/h)^{1/(n+1)}.$$

13. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.

14. Show that the semi-vertical angle of the right cone of given total surface and maximum volume is $\sin^{-1} \frac{1}{3}$. [*Patna*, 1935]

15. Show that the cone of greatest volume which can be inscribed in a given sphere has an altitude equal to $\frac{2}{3}$ of the diameter of the sphere. Prove, also, that the curved surface of the cone is a maximum for the same value of the altitude. [*Agra*, 1928]

16. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is

$$\tfrac{4}{27}\pi h^3 \tan^2 \alpha.$$ [*Benares*, 1933]

17. Show that the total surface of a right circular cylinder inscribed in a right circular cone cannot have a maximum value if the semi-vertical angle of the cone exceeds $\tan^{-1} \frac{1}{2}$.

18. One corner of a rectangular sheet of paper of width 1 foot is folded over so as to reach the opposite edge of the sheet. Find the minimum length of the crease. [*Allahabad*, 1932]

19. The cost of fuel for running a train is proportional to the square of the speed generated in miles per hour, and costs Rs. 48 per hour at 16 miles per hour. What is the most economical speed, if the fixed charges are Rs. 300 per hour? [*Allahabad*, 1930]

[Let the speed in miles per hour $= v$. Let the cost of fuel per hour in rupees $= kv^2$. Substituting the given values, $k = 3/16$. Let the distance to be run be s miles. Then the time of run $= s/v$ hours. Hence for this trip the cost of fuel in rupees will be

$$\frac{3}{16}v^2\left(\frac{s}{v}\right), \text{ i.e., } \frac{3}{16}sv.$$

Also the "fixed charges" (i.e., salaries, etc.) for this trip are, in rupees, $300\, s/v$.

Hence the total cost for the trip in rupees

$$= \tfrac{3}{16}sv + 300\,sv^{-1} \quad . \quad . \quad . \quad . \quad . \quad (1)$$

For a minimum (or a maximum), $\tfrac{3}{16}s - 300\,sv^{-2} = 0$,

i.e., $v = 40$.

The second differential coefficient of the expression on the right of (1) is evidently positive. Hence this gives the minimum cost.

Thus the most economical speed is 40 miles per hour.]

20. If POP' and DOD' be two conjugate diameters of an ellipse $PDP'D'$ with its centre at O, and from P and D be drawn two perpendiculars to the major axis cutting it at M and N respectively, find the condition so that $PM + DN$ may be a maximum.

[*Allahabad*, 1929]

21. Tangents are drawn to the ellipse $x^2/a^2 + y^2/b^2 - 1 = 0$ and the circle $x^2 + y^2 - a^2 = 0$ at the points where a common ordinate cuts them. Show that if ϕ be the greatest inclination of these tangents, then

$$\tan\phi = \frac{a-b}{2\sqrt{(ab)}}. \qquad [\textit{Allahabad}, 1928]$$

22. A conjuror completely conceals a sphere lying on a table with a right circular conical cover. Find the least surface of the cover in square inches when the radius of the sphere is 3″, taking 7π equal to 22. Find also the semi-vertical angle of the cone.

[*Allahabad*, 1925]

23. A given quantity of metal is to be cast into a half-cylinder, i.e., with a rectangular base and semicircular ends. Show that in order that the total surface area may be a minimum, the ratio of the length of the cylinder to the diameter of its semicircular ends is $\pi/(\pi + 2)$. [*Andhra*, 1937]

24. N is the foot of the perpendicular drawn from the centre O on to the tangent at a variable point P on the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b)$. Prove that the maximum area of the triangle OPN is $(a^2 - b^2)/4$.

Find also the maximum length of PN. [*I. C. S.*, 1931]

25. Find the maximum and minimum values of $(1 - x)^2 e^x$.

Show that $e^x - (1 + x)/(1 - x)$ steadily decreases as x increases from $-\infty$ to $+1$, and that it has one and only one minimum for values of x between $x = 1$ and $x = +\infty$. [*London*, 1932]

26. The parcel post regulations restrict parcels to be such that the length plus the girth must not exceed 6 ft., and the length must not exceed $3\frac{1}{2}$ ft. Determine the parcel of greatest volume that can be sent by post, if the form of the parcel be a right circular cylinder. Will the result be affected if the greatest length permitted were only $1\frac{3}{4}$ feet? [*Patna*, 1935]

27. Show that if $x = a$ is an approximate position for a maximum or minimum of $f(x)$, then

$$f(a) - \{f'(a)\}^2 f''(a)$$

is, in general, a better approximation than $f(a)$ to the maximum or minimum value.

Hence find, correct to 1 in 1000, the minimum value of $2x^4 - 58x^2 + 133x$ which occurs near $x = 3$. [*Math. Tripos*, 1928]

28. Show that the function $x^5(e^x - 1)$ has two stationary values, one at the origin and the other given approximately by

$$x = 5(1 - e^{-5})$$ [*Math. Tripos*, 1932]

29. Show that $(a - a^{-1} - x)(4 - 3x^2)$, where a is a positive constant, has one and only one maximum value and one and only one minimum value. Determine these values, and show that the difference between them is $\tfrac{4}{9}(a + a^{-1})^3$. What is the least value of this difference for various values of a? [*Math. Tripos*, 1933]

30. An ellipse is inscribed in an isosceles triangle of height h and base $2k$ and having one axis lying along the perpendicular from the vertex of the triangle to the base. Show that the maximum area of the ellipse is $\sqrt{3}\,\pi hk/9$. [*Math. Tripos*, 1934]

CHAPTER XIV

INDETERMINATE FORMS. DIFFERENTIALS

14·1. Indeterminate Forms. The limit of $\varphi(x)/\psi(x)$ as $x \to a$ is, in general, equal to the limit of the numerator divided by the limit of the denominator. But when these two limits are both zero, the quotient reduces to the form $0/0$, which is meaningless. In the present chapter we shall consider what should be done in such and similar other cases.

The form $0/0$ is called an *indeterminate form*. Other indeterminate forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞ and ∞^0.

14·11. The form $0/0$. Let $\varphi(x)$ and $\psi(x)$ be functions which are expansible by Taylor's Theorem in the neighbourhood of $x = a$. Also *let* $\varphi(a) = 0$, *and* $\psi(a) = 0$.

Then

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)} = \lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)}.$$

We have $\lim_{x\to a}\dfrac{\varphi(x)}{\psi(x)} = \lim_{x\to a}$

$$\frac{\varphi(a) + (x-a)\varphi'(a) + (1/2!)(x-a)^2\varphi''(a) + \dots + R_1}{\psi(a) + (x-a)\psi'(a) + (1/2!)(x-a)^2\psi''(a) + \dots + R_2},$$

by Taylor's Theorem, where $R_1 = (1/n!)(x-a)^n \varphi^{(n)}\{a + \theta(x-a)\}$, $0<\theta<1$, and R_2 has a similar meaning. Now, since $\varphi(a) = 0$, and $\psi(a) = 0$, we get, after dividing both numerator and denominator by $(x-a)$,

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)} = \lim_{x\to a}\frac{\varphi'(a) + (x-a)\{(1/2!)\varphi''(a) + \dots\}}{\psi'(a) + (x-a)\{(1/2!)\psi''(a) + \dots\}}$$

$$= \lim_{x\to a}\frac{\varphi'(a)}{\psi'(a)},$$

This proves the proposition.

It is easy to see that *if* $\phi'(a), \phi''(a), \dots \phi^{(n-1)}(a)$ *and* $\psi'(a), \psi''(a), \dots \psi^{(n-1)}(a)$ *are all zero, but* $\phi^{(n)}(a)$ *and* $\psi^{(n)}(a)$ *are not both zero, then*

$$\lim_{x\to a}\frac{\phi(x)}{\psi(x)} = \lim_{x\to a}\frac{\phi^{(n)}(x)}{\psi^{(n)}(x)}.$$

NOTE. 1. The existence of differential coefficients of order higher than n need not be assumed. (Cf. § 13·13, note).

2. The proposition of this article is true even when we have ∞ instead of a. For, writing $x = 1/t$, we have, if

$$\lim_{x\to\infty}\phi(x) = 0 \text{ and } \lim_{x\to\infty}\psi(x) = 0,$$

then $\lim_{x\to\infty}\dfrac{\phi(x)}{\psi(x)} = \lim_{t\to 0}\dfrac{\phi(1/t)}{\psi(1/t)}$

$$= \lim_{t\to 0}\frac{\phi'(1/t)\,t^{-2}}{\psi'(1/t)\,t^{-2}}, \text{ by the proposition proved above,}$$

$$= \lim_{t\to 0}\frac{\phi'(1/t)}{\psi'(1/t)} = \lim_{x\to\infty}\frac{\phi'(x)}{\psi'(x)}.$$

IMPORTANT. The student must not differentiate $\phi(x)/\psi(x)$ as a fraction by the rule of § 2·5. The numerator and denominator have to be differentiated separately.

Ex. Find the limit, as x tends to zero, of $(x - \sin x)/x^3$. We have

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{1-\cos x}{3x^2}, \text{ which is of the form } 0/0,$$

$$= \lim_{x\to 0}\frac{\sin x}{6x}, \qquad \text{,,} \quad \text{,,} \quad \text{,,} \quad \text{,,}$$

$$= \lim_{x\to 0}\frac{\cos x}{6} = \tfrac{1}{6}.$$

14·12. Algebraic Methods. In cases where the expansions of the functions involved are known, or some of the limits are known, they may be used either to solve the problem or to shorten the work.

Thus in the last example, when we got $(\sin x)/x$, we could have put down its limit as 1, without further differentiation. Or, we could have solved this problem as follows:

$$\lim_{x\to 0}\frac{x-\sin x}{x^3} = \lim_{x\to 0}\frac{x-(x-x^3/3!+\dots)}{x^3}$$

$$= \lim_{x\to 0} \frac{\frac{1}{6}x^3 - x^5/5! + \dots}{x^3}$$

$$= \lim_{x\to 0} \tfrac{1}{6} - x^2/5! + \dots$$

$$= \tfrac{1}{6}.$$

The student should note that differentiating the numerator or the denominator repeatedly amounts really to finding the coefficients in the expansion of the function concerned by Maclaurin's Theorem; and if the expansion is already known, the work can be shortened.

But in some cases, when, on account of the non-existence of the differential coefficients, Taylor's and Maclaurin's theorems are not applicable, we can still get the result by algebraic methods.

Ex. Find $\lim_{x\to 0} \dfrac{x^{1/2}\tan x}{(e^x - 1)^{3/2}}$.

As $x^{-1/2}$ cannot be expanded in a Taylor's series in the neighbourhood of $x = 0$, the proposition of the last article is not applicable.

But $\lim_{x\to 0} \dfrac{x^{1/2}\tan x}{(e^x - 1)^{3/2}} = \lim_{x\to 0} \dfrac{x^{1/2}\tan x}{(x + x^2/2! + \dots)^{3/2}}$, by the Exponential Theorem

$$= \lim_{x\to 0} \frac{x^{1/2}\tan x}{x^{3/2}\{1 + x/2! + \dots\}^{3/2}}$$

$$= \lim_{x\to 0} \frac{\tan x}{x} \cdot \frac{1}{\{1 + (x/2!) + \dots\}^{3/2}}$$

$$= 1.$$

EXAMPLES

Evaluate the following limits:

1. $\lim_{x\to 1} (\log x)/(x - 1)$.
2. $\lim_{x\to 0}\{1 - \sqrt{(1 - x^2)}\}/x^2$. [*Aligarh*, 1934]
3. $\lim_{x\to 0} (x - \tan x)/x^3$. [*Delhi*, 1936]
4. $\lim_{x\to 0}(e^x - e^{\sin x})/(x - \sin x)$. [*Dacca*, 1937]
5. $\lim_{x\to 0} \{xe^x - \log(1 + x)\}/x^2$. [*Andhra*, 1937]

6. $\lim_{x\to0}(a^x-b^x)/x$.

7. $\lim_{x\to0}(\tan x-x)/(x^2\tan x)$ [*Nagpur*, 1929]

8. $\lim_{x\to0}\{\cosh x+\log(1-x)-1+x\}/x^2$ [*Mysore*, 1937]

9. $\lim_{x\to1}\left(\dfrac{1}{\log x}-\dfrac{x}{\log x}\right)$

10. $\lim_{x\to0}\dfrac{x\cos x-\log(1+x)}{x^2}$ [*Patna*, 1935]

11. $\lim_{x\to0}(\sin 2x+2\sin^2x-2\sin x)/(\cos x-\cos^2x)$

12. $\lim_{x\to0}\log(1-x^2)/\log\cos x$ [*Madras*, 1934]

13. $\lim_{x\to1}\{x\sqrt{(3x-2x^4)}-x^{6/5}\}/(1-x^{2/3})$.

14. $\lim_{x\to0}\dfrac{x\sin(\sin x)-\sin^2x}{x^6}$ [*Benares*, 1932]

15. $\lim_{x\to0}\dfrac{\sin x\sin^{-1}x-x^2}{x^6}$ [*Bombay*, 1936]

14·2. The Form ∞/∞. *Let $\lim_{x\to a}\varphi(x)$ and $\lim_{x\to a}\psi(x)$ be both infinite. Then*

$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{\varphi'(x)}{\psi'(x)}.$$

For,
$$\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}=\lim_{x\to a}\frac{1/\psi(x)}{1/\varphi(x)},$$
which is of the form $\frac{0}{0}$,

$$=\lim_{x\to a}\frac{-\dfrac{\psi'(x)}{\{\psi(x)\}^2}}{-\dfrac{\varphi'(x)}{\{\varphi(x)\}^2}},\quad\text{by § 14·11,}$$

$$=\lim_{x\to a}\left\{\frac{\psi'(x)}{\varphi'(x)}\left(\frac{\varphi(x)}{\psi(x)}\right)^2\right\}$$

$$=\lim_{x\to a}\frac{\psi'(x)}{\varphi'(x)}\cdot\left\{\lim_{x\to a}\frac{\varphi(x)}{\psi(x)}\right\}^2.\quad\ldots\quad(1)$$

Now let $\lim_{x\to a}\{\varphi(x)/\psi(x)\}=\lambda$. . . . . (2)

Three cases arise.

Case I. If λ is not zero or infinite, we can divide both sides of equation (1) by λ^2 and thus obtain the following:

$$\lambda^{-1} = \lim_{x\to a} \frac{\psi'(x)}{\varphi'(x)}.$$

Hence $$\lambda = \lim_{x\to a} \frac{\varphi'(x)}{\psi'(x)}.$$

Case II. If $\lambda = 0$, then, adding 1 to each side of equation (2),

$$\lambda + 1 = \lim_{x\to a} \frac{\varphi(x) + \psi(x)}{\psi(x)}$$

$$= \lim_{x\to a} \frac{\varphi'(x) + \psi'(x)}{\psi'(x)}, \text{ by case I,}$$

$$= \lim_{x\to a} \frac{\varphi'(x)}{\psi'(x)} + 1.$$

Hence $$\lambda = \lim_{x\to a} \frac{\varphi'(x)}{\psi'(x)}.$$

Case III. If $\lambda = \infty$, then

$$\frac{1}{\lambda} = \lim_{x\to a} \frac{\psi(x)}{\varphi(x)} = \lim_{x\to a} \frac{\psi'(x)}{\varphi'(x)}, \text{ by case II.}$$

Hence $$\lambda = \lim_{x\to a} \frac{\varphi'(x)}{\psi'(x)}.$$

Hence, in every case in which $\lim_{x\to a} \varphi(x) = \infty$ and $\lim_{x\to a} \psi(x) = \infty$, we have

$$\lim_{x\to a} \frac{\varphi(x)}{\psi(x)} = \lim_{x\to a} \frac{\varphi'(x)}{\psi'(x)}.$$

Although the quotient of two functions can at once be put in a form which would give rise to the indeterminate form $0/0$, instead of ∞/∞, and vice versa, care should be taken to select the form which would enable us to evaluate the limit most quickly. Also, in most cases coming under the form ∞/∞, it is necessary to transform to the form $0/0$ as soon as convenient; otherwise the process of differentiating the numerator and denominator would never terminate. Thus, if we once have x^{-1} in the numerator (or the denominator) and the limit when $x \to 0$ has to be found, successive differentiations

would involve $x^{-2}, x^{-3}, x^{-4}, \ldots$, which all $\to \infty$ as $x \to 0$. So a change to the form $0/0$ has to be made at a suitable stage.

NOTE. (i) As before, by writing $x = 1/t$, we can show that the proposition of this article is true even when $a = \infty$.

(ii) The proposition is evidently true also when one or both the limits are $-\infty$.

Ex. Find $\lim_{x \to 0} \dfrac{\log x}{\cot x}$.

This is of the form ∞/∞. We have, therefore,

$$\lim_{x \to 0} \frac{\log x}{\cot x} = \lim_{x \to 0} \frac{1/x}{-\operatorname{cosec}^2 x} \text{ (form } \infty/\infty)$$

$$= \lim_{x \to 0} \frac{-\sin^2 x}{x} \text{ (form } 0/0)$$

$$= \lim_{x \to 0} \frac{-2 \sin x \cos x}{1} = 0.$$

14·21. The form $0 \times \infty$. This can easily be reduced either to the form $0/0$ or to the form ∞/∞.

Ex. Evaluate $\lim_{x \to 0} x \log x$.

We have $\lim_{x \to 0} x \log x$ (which is of the form $0 \times \infty$)

$$= \lim_{x \to 0} \frac{\log x}{1/x} \text{ (form } \infty/\infty)$$

$$= \lim_{x \to 0} \frac{1/x}{-1/x^2} = \lim_{x \to 0} (-x) = 0.$$

14·22. The form $\infty - \infty$. This can be reduced to the form $0/0$ or ∞/∞. For,

$$\phi(x) - \psi(x) = \frac{\dfrac{1}{\psi(x)} - \dfrac{1}{\phi(x)}}{\dfrac{1}{\psi(x)\,.\,\phi(x)}},$$

and if the left hand side is of the form $\infty - \infty$, the right hand side is of the form $0/0$.

Ex. Evaluate $\lim_{x \to \pi/2} (\sec x - \tan x)$.

$$\lim_{x \to \pi/2} (\sec x - \tan x) = \lim_{x \to \pi/2} \frac{1 - \sin x}{\cos x} \text{ (form } 0/0)$$

$$= \lim_{x \to \pi/2} \frac{-\cos x}{-\sin x} = 0.$$

14·3. The forms 0^0, 1^∞, ∞^0. These can be made to depend upon one of the previous forms. For, since

$$\phi(x) = e^{\log \phi(x)}$$

we have $$\{\phi(x)\}^{\psi(x)} = e^{\psi(x) \log \phi(x)};$$

and thus if $\{\phi(x)\}^{\psi(x)}$ assumes any of the forms 0^0, 1^∞, or ∞^0, we have only to determine the limit of $\psi(x) \log \phi(x)$ by one of the previous methods.

The student sometimes wonders why 1^∞ should be an indeterminate form, seeing that 1 raised to any power gives 1. But he should notice that this only entitles him to conclude that if

$$\lim_{x \to a} \psi(x) = \infty,$$

then $$\lim_{x \to a} 1^{\psi(x)} = 1.$$

When $\phi(x) \to 1$ as $x \to a$, and at the same time $\psi(x) \to \infty$ as $x \to a$, then $\lim \{\phi(x)\}^{\psi(x)}$ might be anything; for the value of $\{\phi(x)\}^{\psi(x)}$ when $\phi(x)$ is not exactly, but only approximately, equal to 1, is not necessarily near to 1 when $\psi(x)$ is large. Consider, for example, the value of $(1 \cdot 01)^{1000000}$. Applying the Binomial Theorem, we see at once that it is far greater than 1.

Ex. Evaluate $\lim_{x \to \infty} (1 + a/x)^x$.

Here $\lim_{x \to \infty} \{x \log (1 + a/x)\}$ (which is of the form $0 \times \infty$)

$$= \lim_{x \to \infty} \frac{\log (1 + a/x)}{1/x} \text{ (form 0/0)}$$

$$= \lim_{x \to \infty} \frac{(1 + a/x)^{-1} \cdot (-ax^{-2})}{-x^{-2}}$$

$$= \lim_{x \to \infty} a(1 + a/x)^{-1} = a.$$

Hence $$\lim_{x \to \infty} (1 + a/x)^x = e^a.$$

14·4. Compound Forms. In case a given function can be broken up into two or more factors, the limit of each of which can be easily found (either by the methods of this chapter or by mere substitution), then the limit of the entire function can be determined by evaluating the limit of each factor separately, provided that the product of these limits is not itself in an indeterminate form. A similar rule would apply in the case of a quotient, sum, difference or power. The student should be on the look-out for recognising cases in which such methods are applicable.

Examples

Evaluate the limits of the following functions for the values of x noted against them:

1. $\dfrac{\log \sin 2x}{\log \sin x}, x \to 0.$ 2. $\dfrac{\cot 2x}{\cot x}, x \to 0.$

3. $\dfrac{e^x}{x^3}, x \to \infty.$ 4. $\dfrac{\log x}{a^x}, x \to \infty.$

5. $\dfrac{\log(1-x)}{\cot \pi x}, x \to 1.$ 6. $\dfrac{\sec \pi x}{\tan 3\pi x}, x \to \frac{1}{2}.$

7. $\dfrac{(1+x)^{1/x} - e}{x}, x \to 0.$ [*Punjab*, 1937]

8. $\dfrac{\log(x-1)}{\cot \pi x} \quad \tan \tfrac{1}{2}\pi x, x \to 1$ from the right.

9. $x \log \sin x, x \to 0.$ 10. $\sin x \,.\, \log x, \; x \to 0.$

11. $(1 - \sin x) \tan x, x \to \frac{1}{2}\pi.$ 12. $2^x \sin(a/2^x), x \to \infty.$ [*Dacca*, 1935]

13. $\dfrac{2}{x^2-1} - \dfrac{1}{x-1}, x \to 1.$ 14. $x \tan x - \frac{1}{2}\pi \sec x, x \to \frac{1}{2}\pi.$ $=-1$

15. $\dfrac{1}{x^2} - \dfrac{1}{\sin^2 x}, x \to 0.$ 16. $\sec x - \dfrac{1}{1 - \sin x}, x \to \frac{1}{2}\pi.$

17. $(a^{1/x} - 1)\, x, x \to \infty.$ 18. $(1 + x^{-2})^x, x \to \infty.$

19. $\left(\dfrac{\tan x}{x}\right)^{1/x}, x \to 0.$ $=1$ $\left(\dfrac{\tan x}{x}\right)^{1/x^2}, x \to 0.$ [*Cal.*, '38]

21. $\left(\dfrac{\tan x}{x}\right)^{1/x^3}, x \to 0.$ 22. $(\sin x)^{2 \tan x}, x \to \frac{1}{2}\pi.$ [*All.*, '30]

23. $x^{x^{-a}}, x \to 0$, where a is an even integer and $x > 0$.

24. $x^{1/(1-x)}, x \to 1.$ [*Benares*, 1931]

25. $\{2(\cosh x - 1)/x^2\}^{1/x^2}, x \to 0.$ [*Agra*, 1935]

26. $e\{e^{1/(x-a)} - 1\} \div \{e^{1/(x-a)} + 1\}, x \to a.$

27. $(a_0 x^m + a_1 x^{m-1} + \dots a_m)^{1/x}, x \to \infty.$

28. $(\log x)^{1/(1-\log x)}, x \to e.$

29. $\{(\log x)/x\}^{1/x}, x \to \infty.$ [*Punjab*, 1930]

30. $(\cos x)^{\cot^2 x}, x \to 0.$ [*Patna*, 1933]

14·5. Infinitesimals. Before the theory of limits had been put on a sound basis, an infinitesimal was defined as a number which was different from zero and yet numerically smaller than any assignable quantity.

But if ϵ is an infinitesimal, one sees no difficulty in assigning a number $\frac{1}{2}\epsilon$; and then, how $|\epsilon|$ would be less than $|\frac{1}{2}\epsilon|$ passes comprehension. Hence, according to the modern theory, the infinitesimal does not exist at all.*

When the expression ' infinitesimal," or "infinitely small quantity", is now-a-days used at all, it is used to denote a *variable which tends to zero.* Such a form of expression, appealing as it does to a mode of thinking which is essentially non-arithmetical, is better avoided†.

If y is a function of x, and δy is the increment in y corresponding to an increment δx in x, then the infinitesimals most generally considered are δx and δy.

14·51. Orders of Infinitesimals. The word "order" is used in connection with infinitesimals in the same sense in which it is used for small quantities (§ 6·41). Thus if δx and δy are infinitesimals, δx and δy are said to be of the *same order* if $\lim_{\delta x\to 0}(\delta y/\delta x)$ is a finite number different from zero.

Again, if δx is regarded as an infinitesimal of order 1, then δy is said to be an infinitesimal of *order n* if $\lim_{\delta x\to 0}\{\delta y/(\delta x)^n\}$ is a finite number different from zero.

If $\lim_{\delta x\to 0}(\delta y/\delta x) = 0$, then δy is said to be an infinitesimal of a *higher order* than δx. If $\lim_{\delta x\to 0}(\delta y/\delta x) = \infty$ or $-\infty$, then δy is said to be an infinitesimal of *lower order than* δx.

In problems in which the limit of the ratio of all the infinitesimals to one of them, say δx, is taken, *all infinitesimals of a higher order than* δx *can be omitted, and any infinitesimal (say* δy*) can be replaced by another which differs from* δy *by an infinitesimal of a higher order.* This is obvious because the ratios to δx of the quantities proposed to be omitted or added will all tend to zero

14·52. Element. The word *element* is generally used in the sense of a small increment. Thus when it is said "Let δs be an element of the arc," what is implied is that the length of the arc up to any point is a function of some variable, say x, and that δs is the increment in s corresponding to an increment δx in x. The context will show whether δs is being treated as an infinitesimal, or merely as a small fixed quantity.

*A perusal of the Historical Note at the end of this book will be very instructive in this connection.

†E. W. Hobson: *The Theory of Functions of a Real Variable*, Vol. I, p. 43 (3rd edition).

14·53. Meaning of dx and dy. We know that

$$\frac{d(\sin x)}{dx} = \cos x,$$

where the $d(\sin x)$ and dx, taken separately, have no meaning. Very often this equation is written as

$$d(\sin x) = \cos x \, dx,$$

in which also $d(\sin x)$ and dx are regarded as having really no separate existence and no meaning.

But those who employ infinitesimals consider dy and dx in the equation $dy = y' \, dx$ to be infinitesimals such that their ratio is y'. Thus they really look upon dy/dx as a fraction.

The infinitesimal dy is then called the *differential* of y and the infinitesimal dx the differential of x.

The value of any differential by itself can never be determined, but the ratio of two differentials may be definite.

For clarity of ideas differentials have been avoided in this book. What has been written above is simply to enable students to follow discussions which use these. It should be remembered that equations expressed by means of differentials are, in general, capable of immediate translation into the language of differential coefficients. For example, the equation

$$ds = \sqrt{(dx^2 + dy^2)}$$

is the same as the equation

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}};$$

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$

is the same as the equation

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt};$$

and
$$d^2(\sin x) = -\sin x \, dx^2$$

is the same as

$$\frac{d^2(\sin x)}{dx^2} = -\sin x.$$

14·54. Perfect Differential. The expression

$$f(x,y)\,dx + \phi(x,y)\,dy \quad . \quad . \quad . \quad . \qquad (1)$$

is called a *perfect differential*, or an *exact differential*, if a function $\psi(x,y)$ exists such that its differential $d\psi$, viz.,

$$\frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy$$

is equal, for all values of dx and dy, to

$$f(x,y)\,dx + \phi\,(x,\ y)\,dy.$$

In order that (1) be an exact differential, it is necessary, therefore, that

$$f(x,\ y) = \frac{\partial\psi}{\partial x},\ \text{and}\ \phi(x,y) = \frac{\partial\psi}{\partial y}.$$

Differentiating these equations with respect to y and x respectively, we see at once that, *in all ordinary cases, in order that $f(x,\ y)\,dx + \phi\,(x,y)\,dy$ be an exact differential it is necessary that*

$$\frac{\partial f}{\partial y} = \frac{\partial \phi}{\partial x}.$$

It can be easily shown that this condition is, in general, also sufficient for (1) being an exact differential.

EXAMPLES ON CHAPTER XIV

1. Evaluate $\quad \lim_{x\to 1} \dfrac{\sqrt{x} - 1 + \sqrt{(x-1)}}{\sqrt{(x^2-1)}}$.

2. Does the limit of $x^{-1}\,e^{1/x}$ as $x \to 0$ exist? Do the limits on the right and on the left exist?

3. Show that $\quad \lim_{x\to 0}\left\{\dfrac{\pi}{4x} - \dfrac{\pi}{2x(e^{2\pi}+1)}\right\} = \dfrac{\pi^2}{8}$.

4. Prove that

$$\lim_{x\to 0} \frac{(1+x)^{1/x} - e + \frac{1}{2}ex}{x^2} = \frac{11e}{24}.$$

[*Nagpur*, 1932]

5. Show that $\quad \lim_{x\to a}\left(2 - \dfrac{x}{a}\right)^{\tan \pi(x/2a)} = e^{2/\pi}$.

6. Find $\quad \lim_{x\to 0}\left(\dfrac{\sinh x}{x}\right)^{1/x^2}$.

7. Find $\quad \lim_{x\to\infty}\left(\dfrac{x^2+2x}{5-3x^2}\right)$. [*I. C. S.*, 1934]

8. Find the values of

(i) $\quad \lim_{x\to 0}\left\{\dfrac{1}{x} - \dfrac{1}{x^2}\log(1+x)\right\}$;

(ii) a and b in order that

$$\lim_{x\to 0} \frac{x(1 + a\cos x) - b\sin x}{x^3}$$

may be equal to 1. [*Allahabad*, 1934]

9. Evaluate the limits

(i) $\lim_{x \to 0} \dfrac{5 \sin x - 7 \sin 2x + 3 \sin 3x}{\tan x - x}$,

(ii) $\lim_{x \to 3} \dfrac{\sqrt{3x} - \sqrt{(12 - x)}}{2x - 3\sqrt{(19 - 5x)}}$. [*Allahabad*, 1932]

10. Evaluate $\lim_{x \to 0}\left(\dfrac{1}{x^2} - \dfrac{1}{x \tan x}\right)$. [*Allahabad*, 1930]

11. Obtain, by Maclaurin's theorem, the first four terms of the expansion of $e^{x \cos x}$ in ascending powers of x. Hence or otherwise obtain the limit of

$$\frac{e^x - e^{x \cos x}}{x - \sin x}$$

as x tends to zero. [*Allahabad*, 1933]

12. Find the limit of

$$\frac{1 - ae^{-x} - be^{-2x} - ce^{-3x}}{1 - ae^{x} - be^{2x} - ce^{3x}}$$

as x tends to zero, in the three cases

(i) $a = 3,\ b = -5,\ c = 4$;

(ii) $a = 3,\ b = -4,\ c = 2$;

(iii) $a = 3,\ b = -3,\ c = 1$. [*Math. Tripos*, 1923]

13. Find the limiting value of $x^n e^{-x}$ as x increases to infinity and of $x^m (\log x)^n$ as x decreases to zero, where m and n are positive integers. [*Math. Tripos*, 1923]

14. Prove that

$$\lim_{x \to 1} \frac{1 - 4 \sin^2 \frac{1}{6}\pi x}{1 - x^2} = \frac{\pi\sqrt{3}}{6},$$

and find, correct to 3 decimal places, the value of the root, near to 1, of the equation

$$\frac{1 - 4 \sin^2 \frac{1}{6}\pi x}{1 - x^2} - 1 \cdot 001 \frac{\pi\sqrt{3}}{6} = 0.$$

[*Math. Tripos*, 1932]

15. Evaluate the limit

$$\lim_{x \to 0} (a^x - b^x)/(c^x - d^x),$$

where a, b, c, d are positive and c is not equal to d.

Evaluate also $\lim_{x \to 1} (1 - x + \log x)/\{1 - (2x - x^2)^{1/2}\}$.

[*Math. Tripos*, 1934]

CHAPTER XV

TAYLOR'S THEOREM

15·1. Taylor's Series. We have proved in Chapter VI that

$$f(x) = f(a) + (x - a) f'(a) + (1/2!) (x - a)^2 f''(a) + \dots$$
$$+ \{1/(n-1)!\} (x - a)^{n-1} f^{(n-1)} (a)$$
$$+ (1/n!) (x - a)^n f^{(n)} \{a + \theta(x - a)\}.$$

Denoting the first n terms on the right by $S_n(x)$ and the $(n + 1)$th, i.e., the last, term by $R_n(x)$, we have

$$f(x) = S_n(x) + R_n(x).$$

Suppose now that $f(x)$ possesses differential coefficients of all orders in a domain $(a - \lambda', a + \lambda)$ of x, so that n may be taken as large as we please without violating the conditions under which Taylor's Theorem was established. Suppose further that $R_n(x) \to 0$ as $n \to \infty$, for every value of x in $(a - \lambda', a + \lambda)$.

Let now n increase indefinitely. Then S_n becomes an infinite series and we get

$$\mathbf{f(x) = f(a) + (x - a) f'(a)}$$
$$\mathbf{+ \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^n}{n!} f^{(n)}(a) + \dots,}$$

where $$a - \lambda' < x < a + \lambda.$$

This theorem also is known as Taylor's Theorem.

When we want to distinguish the finite expression of Chapter VI from the infinite series obtained here, the former is referred to as *Taylor's formula*, or *Taylor's development in finite form*, and the latter as *Taylor's series*, or *Taylor's expansion*.

$R_n(x)$ is called the *Remainder after n terms*, because it is equal to the result of subtracting from $f(x)$ the sum of the first n terms of the Taylor's Series of $f(x)$. Various expressions can be found for it. The one found above, viz.,

$$\mathbf{R}_n(\mathbf{x}) = \frac{(\mathbf{x}-\mathbf{a})^n}{\mathbf{n}!}\mathbf{f}^{(n)}\{\mathbf{a}+\theta(\mathbf{x}-\mathbf{a})\},$$

is called Lagrange's form of the remainder.

15·2. Another form for the Remainder. In § 6·9, in the assumed expression for $F(x)$, if we had written the last term as $-(b-x)Q$ instead of $-(1/n!)(b-x)^n Q$, and carried out the rest of the investigation as before, the expression for $R_n(x)$ would have come out as

$$\mathbf{R}_n(\mathbf{x}) = \frac{(\mathbf{x}-\mathbf{a})^n\,(1-\theta)^{n-1}}{(\mathbf{n}-1)\,!}\,\mathbf{f}^{(n)}\{\mathbf{a}+\theta(\mathbf{x}-\mathbf{a})\}.$$

This gives *Cauchy's form of the remainder.*

Of course, even for the same function, the values of θ in the remainders after n terms in Cauchy's and Lagrange's forms would *not* be the same.

15·3. Some expansions.

1. **sin x.** The formal series obtained in § 6·2 viz.,

$$\sin x = x - x^3/3! + x^5/5! \ldots,$$

is convergent for every finite value of x. Moreover $\sin x$ and all its differential coefficients (which are one of the functions $\pm \sin x$, $\pm \cos x$) are numerically less than or equal to 1, whatever be the value of x.

Hence, for every value of x, $R_n \leqslant (x-a)^n/n\,!$, which $\to 0$ as $n \to \infty$.

Therefore $\sin x = x - x^3/3! + x^5/5! - \ldots$ for every value of x.

2. **cos x.** The expansion of this is similar to that of $\sin x$.

3. **log (1 + x).** The formal series which we get by the method of § 6·2 is convergent only if $-1 < x \leqslant 1$. Hence we consider only these values of x.

I. x positive. Lagrange's form for R_n gives

$$R_n(x) = (-1)^{n-1}(1/n)\left(\frac{x}{1+\theta x}\right)^n,$$

which $\to 0$ as $n \to \infty$, because $\{x/(1+\theta x)\}^n$ is not greater than unity, whatever n may be, when x is positive and not greater than 1.

II. x negative. Cauchy's form gives

$$R_n(x) = (-1)^{n-1}x^n \cdot \frac{1}{1+\theta x}\left(\frac{1-\theta}{1+\theta x}\right)^{n-1},$$

which $\to 0$ as $n \to \infty$, because by supposition $|x| < 1$, so $x^n \to 0$, and the other factors do not tend to infinity.

Hence the formal series is a valid expansion when $-1 < x \leqslant 1$.

4. $(1 + x)^m$. If m is a positive integer, the expansion consists of only a finite number of terms and there is no remainder to be considered.

If m is not a positive integer, we get an infinite series which is convergent if $|x| < 1$ and divergent if $|x| > 1$. We take, therefore, $|x|$ to be less than 1. The case when $|x|$ is equal to 1 is more difficult and will not be considered here.

Cauchy's form for R_n gives

$$\begin{aligned} R_n(x) &= \{1/(n-1)!\}\, x^n (1-\theta)^{n-1} \cdot m(m-1)\ldots \\ &\qquad (m-n+1)(1+\theta x)^{m-n}. \\ &= mx(1+\theta x)^{m-1} \cdot \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \cdot \\ &\qquad \frac{(m-1)\ldots(m-n+1)}{(n-1)!}\, x^{n-1}. \end{aligned}$$

Now $mx(1+\theta x)^{m-1}$ remains finite as $n \to \infty$, because θ always lies between 0 and 1 and so $mx(1+\theta x)^{m-1}$ lies between mx and $mx(1+x)^{m-1}$. Also, since $|x| < 1$, so $|\theta x| < \theta$ and therefore $1 + \theta x > 1 - \theta$ even if θx is negative. Hence the second factor in $R_n(x)$, viz.,

$\{(1-\theta)/(1+\theta x)\}^{n-1}$ is not greater than unity. The last factor, viz.

$$\frac{(m-1)(m-2)\ldots(m-n+1)}{(n-1)!}x^{n-1},$$

must $\to 0$ as $n \to \infty$, because the series whose nth term is this is known to be convergent. Hence $R_n(x) \to 0$ as $n \to \infty$, and we find that the Taylor's series for $(1+x)^m$ is valid for every value of m if $|x| < 1$.

Ex. Is $x^{8/3}$ expansible in the neighbourhood of the point $x = 0$ in Taylor's series?

The second differential coefficient is $\frac{8}{3} \cdot \frac{5}{3} \cdot x^{2/3}$. As the differential coefficient of $x^{2/3}$ at $x = 0$ is non-existent (see § 2·12), it follows that $x^{8/3}$ is *not* expansible in any neighbourhood of $x = 0$ in Taylor's series.

Taylor's Theorem is said to *fail* for such a case.

15·4. **Total Differential Coefficient.** In § 11·13 it was *assumed* that if

$$x = \phi(t),\ y = \psi(t),\ x + h = \phi(t+\tau),\ y + k = \psi(t+\tau),$$

then $$\lim_{\tau \to 0} \frac{f(x+h, y+k) - f(x, y+k)}{h} = \frac{\partial f(x, y)}{\partial x}. \quad . \quad (1)$$

We can now prove this. By the mean value theorem, using $\partial/\partial x$ instead of d/dx, as there are now two variables, and assuming $\partial f/\partial x$ to exist, we have

$$\frac{f(x+h, y+k) - f(x, y+k)}{h} = \frac{\partial f(x+\theta h, y+k)}{\partial x},\ 0 < \theta < 1.$$

Assuming further that $\partial f/\partial x$ is a continuous function of the two variables x and y, and remembering that h and k both $\to 0$ as $\tau \to 0$, we see that

$$\lim_{\tau \to 0} \frac{\partial f(x+\theta h, y+k)}{\partial x} = \frac{\partial f(x, y)}{\partial x},$$

which establishes the truth of (1).

15·5. **Commutative property of partial differential coefficients.** We shall prove that a sufficient condition for the equality of

$$\frac{\partial^2 f}{\partial x \partial y} \text{ and } \frac{\partial^2 f}{\partial y \partial x}$$

is that these differential coefficients be continuous.

Consider the expression

$$\frac{f(x+h, y+k) - f(x, y+k) - f(x+h, y) + f(x, y)}{hk}. \quad (1)$$

Let F(x) stand for $f(x, y+k) - f(x, y)$. . . . . (2)

Then the numerator of (1) is $F(x+h) - F(x)$. Keeping y constant and applying the mean value theorem, we see that

$$F(x+h) - F(x) = h\,F_x(x + \theta_1 h),\ 0 < \theta_1 < 1,$$
$$= h\{f_x(x+\theta_1 h,\ y+k) - f_x(x+\theta_1 h,\ y)\},$$

by (2), where f_x means $\dfrac{\partial f}{\partial x}$.

Thus (1) becomes

$$\{f_x(x+\theta_1 h,\ y+k) - f_x(x+\theta_1 h,\ y)\}/k. \quad . \quad . \quad . \quad (3)$$

Regarding (3) as a function of y, and applying the mean value theorem, the expression (3) becomes

$$kf_{xy}(x+\theta_1 h,\ y+\theta_2 k)/k,\ 0 < \theta_2 < 1,$$

where f_{xy} means $\dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right)$.

So (1) becomes $f_{xy}(x+\theta_1 h,\ y+\theta_2 k)$. . . . . (4)

But if we first let $\phi(y)$ stand for $f(x+h,\ y) - f(x, y)$, and thus apply to (1) the mean value theorem by regarding it first as a function of y (x being kept constant) and then as a function of x, we can show that (1) is the same as

$$f_{yx}(x+\theta_3 h,\ y+\theta_4 k). \quad . \quad . \quad . \quad . \quad . \quad . \quad (5)$$

Thus the expressions (4) and (5) are equal. Taking limits as $h \to 0$ and $k \to 0$, and keeping in mind that the functions (4) and (5) are continuous, we get

$$f_{xy}(x, y) = f_{yx}(x, y),$$

which was to be proved.

Examples on Chapter XV

1. Expand e^x by Taylor's Theorem.

2. Find by Maclaurin's Theorem the first four terms and the remainder after n terms of the expansion of $e^{ax}\cos bx$ in a series of ascending powers of x. [*Nagpur*, 1927]

3. Prove that

$$f(x) = f(0) + xf'(0) + (1/2!)x^2 f''(0) + \ldots + (1/n!)x^n f^{(n)}(\theta x),$$

$0 < \theta < 1$, and state the conditions under which the expansion holds good.

Expand $\cot^{-1} x$ in ascending powers of x. [*Aligarh*, 1935]

4. Show for what values of x and at what differential coefficients Taylor's theorem will fail if

$$f(x) = \frac{(x-a)^{10}(x-b)^{13/2}(x-c)^{23/3}}{(x-d)^5}. \qquad [\textit{Allahabad}, 1927]$$

5. Show that

$$\frac{F(x+h) + F(x-h) - 2F(x)}{h^2} = F''(x+\theta h),$$

where θ is some number between -1 and 1. [*Math. Tripos*, 1925]

6. Writing the mean value theorem as

$$f(b) - f(a) = (b-a)f'(\epsilon), \, a < \epsilon < b,$$

find ϵ if $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$.
[*Math. Tripos*, 1935]

7. If the mean value theorem is

$$f(b) - f(a) = f'(x_1)(b-a),$$

find x_1 when $f(x) = x^3 - 3x - 1$, $a = -11/7$, $b = 13/7$.

Sketch the graph of the function between $x = -2$ and $x = +2$ and indicate on it the geometrical significance of your result.
[*Andhra*, 1937]

8. With the help of the mean value theorem, show that if $x > 0$,

$$\log_{10}(x+1) = \frac{x \log_{10} e}{1 + \theta x},$$

where $0 < \theta < 1$. [*Madras*, 1936]

9. Show that the Maclaurin expansion of e^{-1/x^2} is not valid in any interval, however small. [*Mysore*, 1936]

10. Discuss the applicability of Rolle's theorem in the interval $(0, 2)$ to the function $f(x) = 2 + (x-1)^{2/3}$. Illustrate your answer by a sketch. [*Bombay*, 1936]

MISCELLANEOUS EXAMPLES

1. Find the asymptotes of the curve

$$x^3 + 2x^2y - xy^2 - 2y^3 + x^2 - y^2 - 2x - 3y = 0.$$

[*Patna*, 1937]

2. Find all the asymptotes of the curve

$$3x^3 + 2x^2y - 7xy^2 + 2y^3 - 14xy + 7y^2 + 4x + 5y = 0.$$

[*Agra*, 1937]

3. Show that a curve normally approaches its asymptote on opposite sides and towards opposite ends. Discuss the approach of a curve to a pair of parallel asymptotes.

Find the asymptotes of

$$(x - y)^2 (x - 2y) (x - 3y) - 2a(x^3 - y^3) - 2a^2(x + y) (x - 2y) = 0.$$

[*Bombay*, 1935]

4. Show that the initial line is an asymptote to two branches of the curve

$$r^2 \sin \theta = a^2 \cos 2\theta.$$

5. Show that the points of inflexion upon $x^2y = a^2 (x - y)$ are given by $x = 0$, $x = \pm a\sqrt{3}$. [*Patna*, 1937]

6. A line is drawn through the origin meeting the cardioid $r = a (1 - \cos \theta)$ in the points P, Q and the normals at P, Q meet in C. Show that the radii of curvature at P and Q are proportional to PC and QC.

7. Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $7\frac{13}{17}$. [*Madras*, 1935]

8. Find the curvature of the curve

$$y(1 + x^2) = 2x$$

at the point where y is a maximum, and indicate roughly the shape of the curve. [*Andhra*, 1937]

9. Find if the curve $y = \log x$ is concave or convex upwards.

[*Punjab*, 1932]

10. If the equation to a curve be given in polars $r = f(\theta)$, and if $u = 1/r$, prove that the curvature is given by

$$\left(\frac{d^2u}{d\theta^2} + u\right)\sin^3 \phi,$$

where ϕ is the angle between the radius vector and the tangent at the point (r, θ).

Deduce, or otherwise prove, that the curvature is given by

$$\frac{1}{r} \cdot \frac{dp}{dr}.$$ [*Bombay*, 1936]

11. Find the circle of curvature at the origin for the curve

$$x + y = ax^2 + by^2 + cx^3$$ [*Bombay*, 1937]

12. If n denotes the length of the normal PG, intercepted between the point on the curve and the x-axis, show that

$$\frac{\rho}{n} = -\frac{1 + y'^2}{yy''}.$$

Apply the formula to the curve $y = c \cosh (x/c)$. [*Bombay*, 1936]

13. Trace the curves

(i) $ay^2 = x^2y + x^3$,

(ii) $r \cos 5\theta = a$. [*Bombay*, 1935]

14. Trace the curve $y^2(x + a) = x^2(x - a)$, and show that the tangents at the points of inflexion are

$$5x \pm 3\sqrt{3}y - 4a = 0.$$ [*Radford*]

15. Find the asymptotes of the curve

$$xy^3 = a^3(a + x),$$

and trace the curve, showing that the point $(-a, 0)$, where the curve crosses the axis of x, is a point of inflexion. [*Radford*]

16. If $u = \tan^{-1} (y/x)$, prove that

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$ [*Bombay*, 1936]

17. If $u = (1 - 2xy + y^2)^{-1/2}$, prove that

$$\frac{\partial}{\partial x}\left\{(1 - x^2)\frac{\partial u}{\partial x}\right\} + \frac{\partial}{\partial y}\left(y^2 \frac{\partial u}{\partial y}\right) = 0.$$ [*Delhi*, 1935]

18. If $f(x, y, z)$ is a homogeneous function of the nth degree in x, y and z, prove that

$$x^2 \frac{\partial^2 f}{\partial x^2} + y^2 \frac{\partial^2 f}{\partial y^2} + z^2 \frac{\partial^2 f}{\partial z^2} + 2yz \frac{\partial^2 f}{\partial y \partial z} + 2zx \frac{\partial^2 f}{\partial z \partial x} + 2xy \frac{\partial^2 f}{\partial x \partial y}$$
$$= n(n - 1) f(x, y, z).$$ [*Madras*, 1937]

19. If the sides of a triangle ABC vary in such a way that its circum-radius is constant, prove that

$$\frac{da}{\cos A} + \frac{db}{\cos B} + \frac{dc}{\cos C} = 0.$$ [*Mysore*, 1937]

20. Find dB/dA, where A, B, C, the angles of a triangle, satisfy

$$\Sigma \sin B \sin C = b.$$ [*Andhra*, 1936]

21. If $z = xyf(y/x)$, show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

Show also that if z is a constant,

$$\frac{f'(y/x)}{f(y/x)} = \left\{x\left(y + x\frac{dy}{dx}\right)\right\} \Big/ \left\{y\left(y - x\frac{dy}{dx}\right)\right\}.$$ [*Punjab*, 1934]

22. An equilateral triangle moves so that two of its sides pass through two fixed points. Prove that the envelope of the third side is a circle. [*Patna*, 1937]

23. Find the envelope of circles which pass through the origin and have their centres on the equilateral hyperbola $x^2 - y^2 = a^2$. [*Agra*, 1936]

24. Find the envelope of a family of parabolas of given latus rectum and parallel axes when the locus of their foci is a fixed straight line. [*Bombay*, 1937]

25. If (α, β) is the centre of curvature, show that for the curve $x^{2/3} + y^{2/3} = a^{2/3}$,

$$\alpha = x + 3x^{1/3}y^{2/3},$$
$$\beta = y + 3x^{2/3}y^{1/3},$$

and that the equation of the evolute is

$$(x + y)^{2/3} + (x - y)^{2/3} = 2a^{2/3}.$$ [*Punjab*, 1931]

26. Show that the envelope of the family of straight lines

$$ax \sec \alpha - by \operatorname{cosec} \alpha = a^2 - b^2$$

is the curve

$$(ax)^{2/3} + (by)^{2/3} = (a^2 - b^2)^{2/3}.$$ [*Punjab*, 1930]

27. Find for what values of x the following expression is a maximum and a minimum respectively:

$$2x^3 - 21x^2 + 36x - 20.$$ [*Calcutta*, 1936]

28. Investigate the maxima and minima of the ordinates of the curve

$$y = (x - 2)^6 (x - 3)^5.$$ [*Mysore*, 1937]

29. Determine the values of x for which the function $12x^5 - 45x^4 + 40x^3 + 6$ attains an extremum, and investigate the nature of the extrema. [*Dacca*, 1935]

30. Show that the greatest triangle inscribed in a circle is equilateral. [*Patna*, 1937]

31. In a submarine telegraph cable the speed of signalling varies as $x^2 \log (1/x)$, where x is the ratio of the radius of the core to that of the covering. Show that the greatest speed is attained when this ratio is $1 : \sqrt{e}$. [*Agra*, 1937]

32. Find the maximum value of x^2/e^{x^2}. [*Andhra*, 1936]

33. Find the volume of the greatest right circular cone that can be described by the revolution about a side of a right-angled triangle of hypotenuse 2 feet. [*Madras*, 1936]

34. A sector has to be cut from a circular sheet of metal so that the remainder can be formed into a conical shaped vessel of maximum capacity. Find the angle of the sector. [*Madras*, 1934]

35. A cone is circumscribed about a sphere of given radius R. Show that, when the volume of the cone is a minimum, its altitude is 4R and its semi-vertical angle is $\sin^{-1}\frac{1}{3}$. [*Punjab*, 1930]

36. Prove that the minimum intercept made by the axes on the tangent to the ellipse $x^2/a^2 + y^2/b^2 = 1$ is $a + b$. [*Bombay*, 1935]

37. A circular cylinder is to be inscribed in a given sphere of radius of R. If the total surface of the cylinder, including the two ends, is to be a minimum, show that

$$h^2/R^2 = 2(1 - 1/\sqrt{5}),$$

where h is the height of the cylinder. [*Andhra*, 1937]

38. Given the sum of the perimeters of a square and a circle, show that the sum of their areas is least when the side of the square is double the radius of the circle [*Andhra*, 1936]

39. A thin closed rectangular box is to have one edge n times the length of another edge, and the volume of the box is given to be V. Prove that the least surface S is given by

$$nS^3 = 54(n+1)^2V^2$$ [*Punjab*, 1936]

40. Find the point on the curve $y = e^x$ at which the curvature is a maximum, and show that the tangent at this point forms with the axes of coordinates a triangle whose sides are in the ratio $1 : \sqrt{2} : \sqrt{3}$. [*Punjab*, 1937]

41. The sum of the surfaces of a sphere and a cube is given. Show that when the sum of their volumes is least, the diameter of the sphere is equal to the edge of the cube. [*Delhi*, 1937]

42. Evaluate the following limits :

(i) $\lim_{x\to 1} \dfrac{x^5 - 2x^3 - 4x^2 + 9x - 4}{x^4 - 2x^3 + 2x - 1}$,

(ii) $\lim_{x\to\infty} a^x \sin(b/a^x)$ when $a < 1$, also when $a > 1$. [*Dacca*, 1936]

43. Find the limiting value of $(a^x + x)^{1/x}$ when x tends to zero. [*Patna*, 1937]

44. Evaluate :

(i) $\lim_{x\to 0} \dfrac{e^x + \log\{(1-x)/e\}}{\tan x - x}$,

(ii) $\lim_{x \to \pi/4} (\tan x)^{\tan 2x}$,

(iii) $\lim_{x \to 0} \dfrac{\log(1 + kx^2)}{1 - \cos x}$.

[*Bombay*, 1935-37]

45. Evaluate $\lim_{x \to \infty} \{x \tan(1/x)\}$. [*Andhra*, 1937]

46. Evaluate

(i) $\lim_{x \to 0} \dfrac{\sin 2x + 2\sin^2 x - 2\sin x}{\cos x - \cos^2 x}$.

[*Punjab*, 1933]

(ii) $\lim_{x \to 0} \left\{ \dfrac{\pi x - 1}{2x^2} + \dfrac{\pi}{x(e^{2\pi x} - 1)} \right\}$. [*Punjab*, 1929]

47. Evaluate $\lim_{x \to 0} \{(1/x^2) - \cot^2 x\}$. [*Delhi*, 1936]

48. Evaluate the following limits:

(i) $\lim_{x \to a} \dfrac{a^x - x^a}{x^x - a^a}$, (ii) $\lim_{x \to 0} \left(\dfrac{a^x + b^x}{2}\right)^{1/x}$.

[*Madras*, 1934]

49. Show that, if $\phi''(x) > 0$, then

$$\phi\{\tfrac{1}{2}(x_1 + x_2)\} \leqslant \tfrac{1}{2}\{\phi(x_1) + \phi(x_2)\}.$$ [*Bombay*, 1937]

50. Expand $e^x \sin x$ in powers of x, find the nth term and show that the expansion holds for every finite value of x.

[*Punjab*, 1929]

HISTORICAL NOTE*

Graphs. The method of representing the relation between varying quantities by graphs dates from the 14th century, when it was employed by a Paris professor about 1350. What we now denote by abscissa and ordinate he called "longitude" and "latitude." The same method was used later by many writers. The most important problems which arose in connection with the curves thus obtained were the problems of drawing a tangent to a curve at a given point, of finding where the curve had a maximum or a minimum ordinate, and of determining its area. The differential calculus owes its origin really to the problem of tangents, and the integral calculus (which deals with the process which is the inverse of differentiation) to the problem of areas.

Early History. The *student learns the differential calculus* first, but historically the integral calculus (in a crude form, of course) was the first to be invented. The vintage of the year 1612 was extraordinarily abundant, and the question of the volume of wine casks was referred to the astronomer Kepler. He had already evaluated the area of an ellipse, and he successfully applied his method to the new problem by dividing up the given volume into "infinitely many" thin discs, each "infinitely small" Before these methods were invented by Kepler, there was no other method for evaluating areas or volumes except the method of Euclid and the other Greek mathematicians.

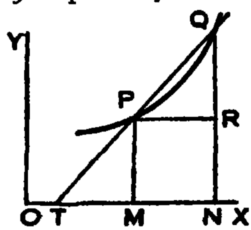
Kepler's methods were taken up by Cavalieri who wrote in 1635 a geometry of "indivisibles," as he called the "infinitely small" elements, and in another book published twelve years later he even succeeded in determining the centres of gravity of solids of variable density.

Many mathematicians solved particular problems of what we now call the integral calculus, and Fermat gave some important generalisations. He came very near to discovering the differential calculus also, because he gave (in 1629) methods for drawing tangents, and discovering maxima and minima which are essentially the same as used to-day.

Methods similar to Fermat began to be used by other mathematicians also. Isaac Barrow published in 1669 his *Lectiones opticæ et geometricæ*, in whch the increments QR, PR of the ordinate and

*Based on an article by A. E. H. Love in the *Encyclopædia Brittanica*, eleventh edition.

abscissa were denoted by a and e. Then the ratio of a to e was found by substituting $x + e$ and $y + a$ for x and y respectively in the equation of the curve, rejecting second order terms, and omitting terms independent of a and e. The process, it is apparent, is equivalent to differentiation. Barrow also noticed the reciprocal relation between the problem of tangents and that of areas.



Barrow thus came nearer even than Fermat to discovering the calculus. But it was reserved for his pupil, the great Newton, and another mathematician Leibnitz, to discover the principles of the calculus.*

Newton. Isaac Newton (1642-1727), as stated above, was Barrow's pupil, and he knew all that Barrow knew to begin with. His original contributions to the subject were contained in three tracts, the first of which was written in 1666. None of these tracts was published till long afterwards.

Newton regarded variable quantities to be generated by the motion of a point. The rate of generation was called a "fluxion." In fact his method comes to this that he regarded x and y as functions of time t. He used $\dot{x}$ to denote what we would denote by dx/dt, and he called it the fluxion of x It was really the velocity of x. By comparing $\dot{y}$ and $\dot{x}$ he could find what we now call the differential coefficient. The method was to denote the "infinitely small" increments of x and y in an "infinitely short" time by xo and yo, and use these in the place of the e and the a of Barrow

Newton gave the rules for what we would now call differentiating a sum of two functions and a function of a function. He proved also the result which we now write as

$$\frac{dx^n}{dx} = nx^{n-1}, \text{ (n integral).}$$

It is not known why the publication of Newton's tracts was so inordinately delayed (the first tract was published 45 years after

*Isolated theorems of the differential calculus had been discovered in India by the Hindu mathematicians much earlier. Thus Mañjula (932 A. D.) knew what we would now write as

$$\delta (\sin \theta) = \cos \theta \, \delta\theta,$$

where $\delta\theta$ is small. Bhāskarācārya (1150 A. D.) used the word *tātkālikagati* (literally instantaneous motion) for the small increment which accrues in a given small interval of time. He was acquainted with the proposition that $\delta f(x) = 0$ if $f(x)$ is a maximum. Some of the later writers had a knowledge of a few other theorems also, but the calculus as such was not studied in India.

it was written), but perhaps Newton saw that his arguments about infinitesimals were not sound. There is reason to think so, for in the *Principia* (1687) he tried to found the calculus of fluxions on the method of limits. Moreover, he used to give rigorous geometrical proofs of theorems in astronomy and mechanics which undoubtedly he discovered first by the method of fluxions.

Leibnitz. Gottfried Wilhelm Leibnitz (1646-1716) was an independent inventor of the calculus. Unlike Newton, he was practically a self-taught mathematician. On the occasion of his first visit to London (1673) he purchased a copy of Barrow's Lectiones. In 1674 he sent an account of the method he had discovered for drawing tangents to curves to Huygens, who sent it to the Royal Society, London.

The foundation of the calculus as we know it to-day was really laid by Leibnitz by writing dy and dx for the differentials of y and x and writing $\int ydx$ for the integral of y. Even in 1675 he wrote $\int ydy = \frac{1}{2}y^2$, just as we write it to-day.

Several letters passed between Leibnitz and Newton from July 1676 to June 1677, when Leibnitz gave Newton a candid account of his differential calculus. Newton did not reply to this, and in his previous letters also he had never disclosed his own methods to Leibnitz. The letters were sent to each other through Oldenberg, the secretary of the Royal Society.

In 1684 Leibnitz published his results in the *Acta eruditorum* in the form of a memoir, and gave methods for finding the differential of a sum (or difference), a product, a quotient and a power, and also the differential of x^m. The rule for the differential of a function of a function was merely illustrated by examples. The rule for maxima and minima was correctly given, utilising the sign of ddy to distinguish between maxima and minima. The method of finding tangents was also given.

Soon the differential calculus (as also the integral calculus) was rapidly developed by Leibnitz and the Bernoullis.

Dispute regarding priority. In 1699 one N. C. de Duillier published a tract in which he stated that Leibnitz had stolen the calculus from Newton. Leibnitz replied to it in the *Acta eruditorum* (1700), citing Newton's letters and the passage in Newton's Principia wherein he had acknowledged Leibnitz to be an independent inventor of the calculus. The dispute was settled at the time. But in 1705, in an anonymous review of a tract of Newton's, a disparaging remark was made against Newton. The insinuation, in fact, was that Newton had stolen the calculus from Leibnitz and had merely changed the notation. Great indignation was aroused, and much was written by the supporters of Newton and Leibnitz which, as Moritz Cantor put it, "redounded to the discredit of all concerned." The dispute continued till several years after the death of Leibnitz.

Later developments. A result of this controversy was that to the British mathematicians it became a point of honour not to use the methods of Leibnitz. They stuck to the method and notation of fluxions, which was not powerful enough. The only advances made by them in the 18th century were the discoveries by Brook Taylor and Colin Maclaurin of the theorems that go by their names.

On the continent, on the other hand, rapid development took place, and partial differentiation was soon introduced (1734) by Euler.

Criticism of the Calculus. Several mathematicians, including Huygens, had opposed the method of the calculus from the very beginning. A Dutch physician (B. Nieuwentijt) attacked the method on account of the use of quantities which are at one stage of the process treated as somethings and at a later stage as nothings. In England Bishop Berkeley attacked (in 1734) the vague conception of quantities which were neither zero, nor different from zero, but in some intermediate stage. He alleged that religion was no worse than mathematics, seeing that things like these had to be accepted on mere faith. The controversy which ensued made it plain that infinitesimals must be discarded. But progress was slow. It was not until 1823, when Cauchy published his treatise on the differential calculus, that the whole thing was put on a satisfactory basis. But even after this much remained to be done. The trend of modern developments has been to show that many theorems break down in exceptional circumstances, and much of the research work of the last fifty years or more has been devoted to the discovery of forms in which the theorems have no exception and yet are as general as possible.

ANSWERS TO THE EXAMPLES

PAGES 4-5

1. (i) Continuous var.; (ii) con. var.; (iii) no; (iv) no.
2. (i) none; (ii) 0; (iii) none; (iv) $\pi/2 \pm n\pi$; (v) 0; (vi) 1, 2.
5. $\sin^{-1}x$, $\log x$, $\sqrt{x}$, ... 6. Yes.

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1. $a^6 + 2a^2 + 1$. 3. $x = 1$ and $x = 3$. 4. 1.
5. No. 7. Continuous. 9. Integral values.
10. $x = 0, \frac{1}{2}, 1$. 11. (i) No; (ii) yes; (iii) no.

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1. $1, 5x^4, -4x^{-5}$. 2. $\frac{1}{2}x^{-1/2}, \frac{2}{3}x^{-1/3}, -\frac{5}{6}x^{-11/6}$.
3. $\frac{1}{2}x^{-1/2}, \frac{3}{2}x^{1/2}, -\frac{3}{2}x^{-5/2}$. 4. $4x, 18x^5, -35x^{-8}$.
5. $8e^x, \sqrt{2}\cos x$. 6. $-3/x, -4x^4$. 7. $12x^2 + 3\cos x$.
8. $-5\sin x - 2e^x$. 9. $6/x - \frac{1}{2}x^{-1/2}$. 10. nx^{n-1}.
11. $2(ax + b)$. 12. $ma^m x^{m-1}$. 13. $x^2 + 3x^2 \log x$.
14. $e^x \cos x + e^x \sin x$. 15. $x^{-1}\cos x - \sin x \log x$.
16. $(1/x)\log_a e + a/x$. 17. $(e^x/x)\log_a e + e^x \log_a x$.
18. $(\sin x)(1/x)\log_a e + \cos x \cdot \log_a x$.
19. $15x^4 e^x + 3x^5 e^x$. 20. $(9/7)\, e^x (\sin x + \cos x)$.
21. $(8/\sqrt{x}) + (4/\sqrt{x})\log x$. 22. $1 + x + (x^2/2!) + (x^3/3!) + \dots$

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1. $(n x^{n-1} \cdot \log x - x^{n-1})/(\log x)^2$.
2. $(nx^{n-1}\log_a x - x^{n-1}\log_a e)/(\log_a x)^2$.
3. $\{-\sin x \log x - (1/x)\cos x\}/(\log x)^2$.
4. $\{2ax(\sin x + \cos x) - (\cos x - \sin x)(ax^2 + b)\}/(\sin x + \cos x)^2$.
5. $\{\frac{1}{2}x^{-1/2}(a^{1/2} - x^{1/2}) + (a^{1/2} + x^{1/2})(\frac{1}{2}x^{-1/2})\}/(a^{1/2} - x^{1/2})^2$.

6. $\{(\sec^2 x)(5x+7) - (3+\tan x) \cdot 5\}/(5x+7)^2$.

7. $\{(\sec^2 x - \operatorname{cosec}^2 x)\log x - (1/x)(\tan x + \cot x)\}/(\log x)^2$.

8. $\{-\operatorname{cosec}^2 x\,(x+e^x) - (1+e^x)\cot x\}/(x+e^x)^2$.

9. $\{(e^x + \sec^2 x)(\cot x - x^n) + (\operatorname{cosec}^2 x + nx^{n-1})(e^x + \tan x)\} \div (\cot x - x^n)^2$.

10. $\{(10x+6)(2x^2+3x+4) - (4x+3)(5x^2+6x+7)\} \div (2x^2+3x+4)^2$.

11. $-\cos x/\sin^2 x$. 12. $\sin x/\cos^2 x$.

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1. $e^{x^3} \cdot 3x^2$, $(\cos x^3)\,3x^2$, $-(\sin x^3)(3x^2)$, $(\sec^2 x^3)(3x^2)$, $(-\operatorname{cosec}^2 x^3)(3x^2)$, $3/x$.

2. $e^{3x} \cdot 3$, $3\sin^2 x \cdot \cos x$, $3\cos^2 x \cdot (-\sin x)$, $3\tan^2 x \cdot \sec^2 x$, $-3\cot^2 x \cdot \operatorname{cosec}^2 x$, $3(\log x)^2 \cdot (1/x)$.

3. $3e^{3x}$, $3\cos 3x$, $-3\sin 3x$, $3\sec^2 3x$, $-3\operatorname{cosec}^2 3x$, $1/x$.

4. $nx^{n-1}/(x^n + a)$, $e^x/(e^x+1)$, $\cos x/(\sin x + 1)$, $-\sin x/\cos x$, $\cot x \cdot \sec^2 x$, $-\tan x \operatorname{cosec}^2 x$, $1/x\log x$, $(1/\sin x)\log_a e \cdot \cos x$.

5. $5e^{5x}$, $e^{(1+\log x)}/x$, $e^{\sin x}\cos x$, $-e^{\cos x}\sin x$, $e^{\tan x}\sec^2 x$, $-e^{\cot x}\operatorname{cosec}^2 x$.

6. $nx^{n-1}\cos x^n$, $\cos(\log x)/x$, $\cos e^x \cdot e^x$, $-\cos(\cos x) \cdot \sin x$, $\cos(\tan x) \cdot \sec^2 x$.

7. $-n\,x^{n-1}\sin x^n$, $-\sin(\log x)/x$, $-\sin e^x \cdot e^x$, $\sin(\cos x) \cdot \sin x$, $-\sin(\tan x) \cdot \sec^2 x$.

8. $5x^4\sec^2 x^5$, $\sec^2(\log x)/x$, $\sec^2 e^x \cdot e^x$, $\sec^2(\sin x) \cdot \cos x$.

9. $\cos x/2\sqrt{(\sin x)}$, $1/2x\sqrt{(\log x)}$, $-\sin x/2\sqrt{(\cos x)}$, $(\sec^2 x)/2\sqrt{(\tan x)}$, $-\operatorname{cosec}^2 x/2\sqrt{(\cot x)}$.

10. $-(\sin x)^{-2}\cos x$, $-(\log x)^{-2}/x$, $(\cos x)^{-2}\sin x$, $-(x^n+a^n)^{-2} \cdot nx^{n-1}$, $-\frac{1}{2}(x+a)^{-3/2}$.

11. $na(ax+b)^{n-1}$, $a/(ax+b)$, ae^{ax+b}, $a\cos(ax+b)$, $a\sec^2(ax+b)$.

12. $8x\cos x^2 + (5\cos x)/(5\sin x + 6)$, $e^x\sec^2 e^x - 12ax^3/(ax^4+b)$.

13. $-(\sin\sqrt{x})(\frac{1}{2}x^{-1/2})\log\sin x + (\cos\sqrt{x})\cos x/\sin x$, $-4\cos^3 x \cdot \sin x \cdot \cos x^4 - \cos^4 x \cdot 4x^3 \cdot \sin x^4$, $\cos x \cdot e^{\sin x} \cdot \sin e^x + e^{\sin x} \cdot \cos e^x \cdot e^x$.

14. $m(x+a)^{m-1}(x+b)^n + n(x+a)^m \cdot (x+b)^{n-1}$, $2mx(x^2+a)^{m-1}(x^2+b)^n + 2nx(x^2+a)^m(x^2+b)^{n-1}$, $npx^{n-1}(x^n+a)^{p-1}(x^m+b)^q + mqx^{m-1}(x^n+a)^p(x^m+b)^{q-1}$.

15. $\{(-3x^2 \operatorname{cosec}^2 x^3)(ax+b) - a\cot x^3\}/(ax+b)^2$,
$\{(3\tan^2 x \,.\, \sec^2 x)(ax^2+b) - 2ax\tan^3 x\}/(ax^2+b)^2$,
$\{-\tan x \,.\, \tan\log x - \log\cos x \,.\, \sec^2\log x \,.\, (1/x)\}/\tan^2\log x$,
$\{e^{\sin x} \,.\, \cos x \,.\, \sin x^n - e^{\sin x} \,.\, \cos x^n \,.\, (nx^{n-1})\}/\sin^2 x^n$,
$\{\frac{1}{2}(\sin x)^{-1/2} \,.\, \cos x \,.\, \sin\sqrt{x} - \sqrt{(\sin x)} \,.\, \cos\sqrt{x} \,.\, \frac{1}{2}x^{-1/2}\}$
$\div (\sin\sqrt{x})^2$.

16. $\{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})\}/(e^x - e^{-x})^2$,
$(2ax+b)/(ax^2+bx+c) + \cot x + nx^{n-1}/(x^n+a^n)$,
$a/(ax+b) - p/(px+q)$.

17. $nx^{n-1}f'(x^n)$, $anx^{n-1}f'(ax^n+b)$, $\cos x\, f'(\sin x)$, $\sec^2 x\, f(\tan x)$.

18. $\log x + 1$, $-\sin x\log\sin x + \cos x \,.\, \cot x$,
$\sec^2 x\log(ax+b) + (\tan x)\{a/(ax+b)\}$.

19. $\frac{1}{3}(a+bx^2)^{-2/3} \,.\, 2bx$, $\frac{1}{2}m\,(a+bx)^{(m-2)/2} \,.\, b$,
$\frac{1}{2}(a+bx^m)^{-1/2} \,.\, mbx^{m-1}$.

20. $e^{a+bx-cx^2}(b-2cx)$, $\{na_0 x^{n-1} + (n-1)a_1x^{n-2} + \ldots + a_{n-1}\}$
$\div (a_0x^n + a_1x^{n-1} + \ldots + a_n)$.

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1. $(1/\sin^{-1} x^4)\{1/\sqrt{(1-x^8)}\} \,.\, 4x^3$,
$(1/\cos^{-1} x^4)\{-1/\sqrt{(1-x^8)}\} \,.\, 4x^3$,
$(1/\tan^{-1} x^4)\{1/(1+x^8)\} \,.\, 4x^3$,
$(1/\sec^{-1} x^4)\{1/x^4\sqrt{(x^8-1)}\} \,.\, 4x^3$.

2. $4(\sin^{-1} x^4)^3\{1/\sqrt{(1-x^8)}\} \,.\, 4x^3$,
$n(\cos^{-1} x^4)^{n-1}\{-1/\sqrt{(1-x^8)}\} \,.\, 4x^3$,
$2(\tan^{-1}\sqrt{x})\{1/(1+x)\}\frac{1}{2}x^{-1/2}$,
$\frac{1}{3}(\cot^{-1} x^3)^{-2/3}\{-1/(1+x^6)\} \,.\, 3x^2$.

3. $a^{\sin 2x} \,.\, \log a \,.\, \cos 2x \,.\, 2$, $a^{\tan 5x} \,.\, \log a \,.\, \sec^2 5x \,.\, 5$,
$a^{\sec nx} \,.\, \log a \,.\, \sec nx \,.\, \tan nx \,.\, n$,
$e^{\operatorname{cosec}(\sin x)}\{-\operatorname{cosec}(\sin x) \,.\, \cot(\sin x)\}\cos x$

4. $\{\cos\log(x^2+1)\}\{1/(x^2+1)\} \,.\, 2x$, $\{1/(e^{4x+2}+1)\} \,.\, e^{2x+1} \,.\, 2$,
$-\{1/a^{bx+c}\sqrt{(a^{2bx+2c}-1)}\} \,.\, a^{bx+c} \,.\, \log a \,.\, b$,
$\sec(a^x+x^a)^2 \,.\, \tan(a^x+x^a)^2 \,.\, 2(a^x+x^a) \,.\, (a^x\log a + ax^{a-1})$.

5. $\frac{1}{2}(\log\sin x)^{-1/2} \,.\, \operatorname{cosec} x \,.\, \cos x$,
$\frac{1}{2}(\sec\sqrt{x})^{-1/2}(\sec\sqrt{x} \,.\, \tan\sqrt{x}) \,.\, \frac{1}{2}x^{-1/2}$,
$\frac{1}{2}(\sin^{-1} x^5)^{-1/2}\{1/\sqrt{(1-x^{10})}\} \,.\, 5x^4$,
$\frac{1}{2}(\cot^{-1} e^x)^{-1/2}\{-1/(1+e^{2x})\} \,.\, e^x$.

6. $m \sin^{m-1} nx \,.\, \cos nx \,.\, n$, $2 \cot (3e^x + 1) \,.\{- \operatorname{cosec}^2 (3e^x + 1)\} 3e^x$,
$3 \operatorname{cosec}^2(m \sin^{-1} x) \,.\, \{- \operatorname{cosec}(m \sin^{-1} x) \cot(m \sin^{-1} x)\} \,.\, m \div \sqrt{(1 - x^2)}$.

7. $1/\{1 + x^2/(1 + x^2)\}\{\sqrt{(1 + x^2)} - x(1 + x^2)^{-1/2}. x\}/(1 + x^2)$,
$[-1/\sqrt{\{1 - (x - x^{-1})^2/(x + x^{-1})^2\}}]. \{(1 + x^{-2})(x + x^{-1}) - (1 - x^{-2})(x - x^{-1})\}/(x + x^{-1})^2$.

8. $n \operatorname{cosec}^{n-1} x^m \,.\, (- \operatorname{cosec} x^m \cot x^m) \,.\, mx^{m-1}$,
$n(\operatorname{cosec}^{-1}x^m)^{n-1}\{-1/x^m\sqrt{(x^{2m} - 1)}\}mx^{m-1}$,
$n \sec^{n-1}(ax^2 + bx + c)\{\sec(ax^2 + bx + c)\tan(ax^2 + bx + c)\} \times (2ax + b)$.

9. $(1/\sin^{-1} e^{3x})\{1/\sqrt{(1 - e^{6x})}\}. e^{3x} \,.\, 3$,
$(1/\cot^{-1} a^{5x+3})\{- 1/(1 + a^{10x+6})\}a^{5x+3} \,.\, \log a \,.\, 5$,
$\{1/\sec (ax + b)^3\}\{\sec (ax + b)^3 \tan (ax + b)^3\} \,.\, 3a (ax + b)^2$.

10. $10^{10^x} . \log_e 10 \,.\, 10^x \,.\, \log_e 10$, $[1/\sqrt{\{1 - (1 + x^2)^{-1}\}}]. (-\frac{1}{2})\times (1 + x^2)^{-3/2} \,.\, 2x$, $(1/\cosh x) \sinh x$, $(1/\log \log x^2)(1/\log x^2)(2/x)$.

11. $a^{\text{arc sin } x} \log a$, $2e^t\sqrt{t}$, $\frac{1}{2} (\log_{10} e) / x^2$.

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1. $x^x (1 + \log x)$, $x^{\sin x} (\sin x \,.\, x^{-1} + \cos x \log x)$,
$x^{\sin 2x}\{(\sin 2x)/x + 2 \cos 2x \,.\, \log x\}$,
$x^{\cos ax}\{(\cos ax)/x - a \sin ax \log x\}$,
$x^{\cot bx}\{(\cot bx)/x - b \operatorname{cosec}^2 bx \log x\}$, $5x^{5x^3+2} \log ex^3$.

2. $(1 - x^2)^{-3/2} \cos^{-1} x - x/(1 - x^2)$, $xe^x \sin x (\cot x + 1 + 1/x)$.

3. $(\log x)^{\sin x} \{ (\sin x)/(x \log x) + \cos x \log \log x)\}$,
$(\sin x)^{\log x}\{(\log x) (\cot x) + (1/x) \log \sin x\}$,
$(\sin^{-1} x)^{\log x}\{(\log x)/(\sin^{-1} x) (1 - x^2)^{1/2} + (1/x) \log \sin^{-1}x\}$
$(1/x) (\operatorname{cosec}^{-1} x)^{\log x}\{- (\log x) /(\operatorname{cosec}^{-1} x)\sqrt{(x^2 - 1)} + \log \operatorname{cosec}^{-1} x\}$.

4. $(\text{vers}^{-1} x)^{\cos x}$ $(\cos x)/(\text{vers}^{-1} x) (2x - x^2)^{1/2} - \log \text{vers}^{-1} x \,.\, \sin x\}$,
$(\operatorname{cosec}^{-1} x)^{ax+b}\{- (ax + b)/(\operatorname{cosec}^{-1} x) x\sqrt{(x^2 - 1)} + a \log \operatorname{cosec}^{-1} x\}$,
$(\tan^{-1} x)^{(\cos x + \sin x)}\{(\cos x + \sin x)/(\tan^{-1} x) (1 + x^2) + (\cos x - \sin x) \log \tan^{-1} x\}$

5. $(1 + x^{-1})^x\{\log (1 + x^{-1}) - 1/(1 + x)\} + x^{(1/x)-1}\{(1 + x) - \log x \}$,

$(\cot x)^{\sin x}\ (\cos x \log \cot x - \sec x) + (\tan x)^{\cos x}\ \{\operatorname{cosec} x - \sin x \log \tan x\}.$

6. $x^3 (x^2 + 4)^{1/2} (x^2 + 3)^{-1/2}\{3/x + x/(x^2 + 4) - x/(x^2 + 3)\},$
$\frac{1}{2}(x - a)^{1/2} (x - b)^{1/2} (x - p)^{-1/2} (x - q)^{-1/2}\{1/(x - a) + 1/(x - b) - 1/(x - p) - 1/(x - q)\},$
$x^4\sqrt{(1 + \tan x)} \,.\, \sec^2 x.\{(4/x) + \frac{1}{2} \sec^2 x/(1 + \tan x) + 2 \tan x\}.$

7. $(x - 1)^2 (x + 2)^3 (x + 4) \log x \,.\, \{2/(x - 1) + 3/(x + 2) + 1/(x + 4) + 1/x \log x\}$
$(\sin x)(\log x)x^x \cos x\{\cot x + 1/x \log x + 1 + \log x - \tan x\}.$

8. $(1 - 2x)^{1/2} (1 + x)^{1/2} \sec^2 ax\{1/(2x - 1) + 1/2(1 + x) + 2a \tan ax\},$
$3^x \,.\, x^{5+x} \,.\, \cos^{-1} x \,.\, \{\log 3 + \log x + (5 + x)/x - 1/(\cos^{-1} x)\sqrt{(1 - x^2)}\}.$

PAGE 42

1. $-(x^2 + ay)/(ax + y^2).$ 2. $-(x/y)^{n-1}.$

3. $y\, x^{y-1} / \{1 - x^y \log x\}.$

4. $-\{2x^{-3/5} + 3x^{-4/5} y^{1/5}\}\{3x^{1/5} y^{-4/5} + 2y^{-3/5}\}^{-1}.$

5. $-y(y + x \log y)/x(x + y \log x).$

6. $y\sqrt{(1 - y^2)}\{1 - \sqrt{(1 - x^2)}\, e^x \log y\} \div \sqrt{(1 - x^2)} \,.\, \{e^x\sqrt{(1 - y^2)} - y\}.$

7. $-\{yx^{y-1} + y^x \log y\}/\{x^y \log x + xy^{x-1}\}.$

8. $\{\sin y \tan x\, (\cos x)^{\sin y} - \cot x \cos y\, (\sin x)^{\cos y}\} \div \{(\cos x)^{\sin y} \cos y \log \cos x - (\sin x)^{\cos y} \sin y \log \sin x\}$

9. $\{y - (x^2 + y^2)^{1/2} (2x^2 + y^2)\}/\{xy\, (x^2 + y^2)^{1/2} + x\}.$

10. $x\{2\sqrt{(x^2 + y^2)} - x^2 + y^2\}/y\{2\sqrt{(x^2 + y^2)} + x^2 - y^2\}.$

11. $\cot \frac{1}{2}t.$ 12. $\tan t.$ 13. $t\,(e^t - \sin t)/(1 + t \cos t).$

14. $\{\cos t \,.\, \sqrt{(1 - t^2)} - 2\}/\{3t^2 (\cos t^3 - \sin t^3)\sqrt{(1 - t^2)}\}$

PAGE 43

1. $(2x + 1)/2\,(x^2 + x + 1),\ -2(8x + 13)/3\,(4x + 5)\,(2x + 1).$

2. $2x - 10x\,(x^2 + 1)^{-2},\ 3 - (11x^2 + 16x + 11)/(x^2 - 1)^2.$

3. $3a/(x^2 + a^2),\ 1/\sqrt{(a^2 - x^2)},\ a/(a^2 + x^2) + 2a/(a^2 + 4x^2).$

4. $-3/\sqrt{(1 - x^2)},\ -1/2\sqrt{(1 - x^2)},\ 3/\sqrt{(1 - x^2)}.$

5. $-1/(1 + x^2),\ 1/2(1 + x^2).$

6. $1/\sqrt{(1 - x^2)} - 1/2\sqrt{(x - x^2)}.$ 7. $(1 - x^2)^{-3/2}.$

Page 47

1. $2\cos 2x$, $3x^2\cos x^3$, $e^x\cos e^x$, $\cot x$, $\frac{1}{2}\cos x/\sqrt{(\sin x)}$.
2. $(-x^4 - 9x^2 + 10x)/(x^3+5)^2$,
   $-(x\sin x\log x + \cos x)/x(\log x)^2$,
   $(x - \sin 2x)/x^3\cos^2 x$, $e^{2x}(2x\log x - 1/x)(\log x)^2$.
3. $e^{\tan x}\sec^2 x$, $3x^2\sin x + x^3\cos x$,
   $\cos 2x$, $-(1/x)\sin\log x^2$.
4. (i) $1/\{\sin^{-1}x\,.\,\sqrt{(1-x^2)}\}$, (ii) $-(\sin\log x)/x$.
5. $1/2\sqrt{x}$, $\sin 2x$, $e^{\sqrt{x}}/2\sqrt{x}$.
6. $(1/x)\log_a e$, $a/2\sqrt{x}$, $1/(1+x^2)$, $-x/\sqrt{(a^2-x^2)}$.

Pages 49-51

1. $nx^{n-1}\sin ax + ax^n\cos ax$. 2. $-(2/x)\sin(4\log x)$.
3. $1/(1+x^2)\tan^{-1}x$. 4. $(1/x)\log_{10}e$.
5. $-\frac{1}{2}(x^2+2bx+a^2)(a^2-x^2)^{-1/2}(b+x)^{-3/2}$.
6. $(a^2-x^2)^{-3/2} + 3x^2(a^2-x^2)^{-5/2}$.
7. $e^{x^2}(1+x^2)^{-3/2}\{x(1+2x^2)\tan^{-1}x + 1\}$.
8. $(\log_a e)\{1/(x-1) - x/2(x^2+1)\}$.
9. $1/2\sqrt{\{(x-a)(x-b)\}}$.
10. $\frac{1}{2}e^{\sqrt{x}+2}/\sqrt{x} - \frac{1}{2}e^{\sqrt{(x+2)}}/\sqrt{(x+2)}$.
11. $a^{\log\log x}.\,(\log a)/x\log x + b^{x\log x}.\,(\log b)\,.\,(1+\log x)$.
12. $1/x\sqrt{(x+1)}$.
13. $ab/(a^2\cos^2 x + b^2\sin^2 x)$.
14. $(\pi/180)\sec x^\circ\tan x^\circ$.
15. $\{\log\sin x + x\cot x\log x\}/2x\sqrt{(1+\log x\,.\,\log\sin x)}$.
16. $4/(1+x^2+x^4)$.
17. $(1-2xt\cos x)/2xt(t-\sin x)$, where $t^2 = 1+\log x$.
18. $\cot x\,.\,10^{\log\sin x}\,.\,\log_e 10$. 19. $2(\log_e 7)(x+1)\,7^{x^2+2x}$.
20. $-\operatorname{cosec}^2 x\coth x - \cot x\operatorname{cosech}^2 x$.
21. $(\tan x)^{\log x}\{2\operatorname{cosec}2x\log x + (1/x)\log\tan x\}$
    $+ (\cot x)^{\sin x}\{\cos x\log\cot x - \sec x\}$.
22. $(1-x^2)^{-3/2}$.
23. $\{1-2x\sqrt{x}\}/2\{\sqrt{x} + x\sqrt{x} + 2x^2 + x^3\sqrt{x}\}$.
24. $\sin\alpha/(1-2x\cos\alpha + x^2)$.
25. $(\sqrt{2})(1+x^2)/(1+x^4)$. 26. $e^{ax}(a\sin bx + b\cos bx)$.
27. $-3\cot^2(e^{3x}x^x)\operatorname{cosec}^2(e^{3x}x^x)\,.\,e^{3x}x^x(4+\log x)$.

28. $\log x \,.\, \log\log x + \log\log x + 1$.

29. $\cos x \sin 2x \sin 3x \sin 4x + 2 \cos 2x \sin x \sin 3x \sin 4x$
$+ 3 \cos 3x \sin x \sin 2x \sin 4x + 4 \cos 4x \sin x \sin 2x \sin 3x$.

30. $-\frac{1}{2}$. **31.** $x^{\sin^{-1} x}\{(1/x) \sin^{-1} x + (\log x)/\sqrt{(1 - x^2)}\}$.

32. $3/\sqrt{(2e^{-3x} - 1)}$. **33.** $(1 - x^2)^{-3/2}$.

34. $(1/x)\, x^{(\log x)^{\log\log x}} . (\log x)^{\log\log x} . \{1 + 2 \log\log x\}$.

35. $x^{(x+2x)}\{(1/x) + (\log x)(1 - \log x)\}$. **36.** $x\,(x^x)^x \log (ex^2)$.

37. $x^{\tan^{-1} x}\{x^{-1} \tan^{-1} x + (1 + x^2)^{-1} \log x\}/\{1 + x^{2 \tan^{-1} x}\}$.

38. $-amb/[1 + \{m \tan^{-1} (bx)\}^2]\,(1 + b^2 x^2)$.

39. $e^x \cos e^x/(1 + \sin^2 e^x)$. **40.** $nx^{n-1} e^{x^n}$.

41. $3/2(1 + t)\sqrt{t}$. **42.** $1/x \log x \log^2 x \log^3 x \,..\, \log^{n-1} x$.

43. $u^2 v^3 w\,(2u'/u + 3v'/v + w'/w)$, $u^v\,(v\,u'/u + v' \log u)$;
$v(\log_w u)^{v-1} (\log_e w)^{-2}\{(u'/u) \log_e w - (w'/w) \log_e u\}$
$+ (\log^w u)^v (\log_e \log_w u) v'$.

47. $2x\,(6x^2 - y)/(x^2 - 6y^2)$. **48.** $-(ax + hy + g)/(hx + by + f)$.

49. $\{y^{\cot x} (\log y) \operatorname{cosec}^2 x - y\,(\tan x)^{y-1} \sec^2 x\}$
$\div \{(\tan x)^y \log \tan x + \cot x \,.\, y^{(\cot x - 1)}\}$.

50. $my/\{[1 + \{m \tan^{-1} (y/x)\}^2](x^2 + y^2) + mx\}$.

51. $-\tan t$. **52.** $t(2 - t^3)/(1 - 2t^3)$. **55.** $x(1 - x^4)^{-1/2}$.

56. $1/3\,(1 + x^{2/3})\,x^{2/3}$. **57.** $\frac{1}{2}$.

58. $-(\log x)^{\tan x}\{(\tan x/x \log x) + \log\log x \sec^2 x\}$
$- m \cos (m \cos^{-1} x)\,(1 - x^2)^{-1/2}$.

59. 1. **60.** $b/(ab + 2ay)$. **61.** $-(y^2 \tan x)/(1 - y \log \cos x)$.

62. $y\{1/(1 + x) - 2x/(2 - x^2) + 3x^2/(3 + x^3) - 4x^3/(4 - x^4) + \ldots\}$

63. $(\sec^2 x)/(2y - 1)$.

Page 58

1. $a_1 + a_2 \cos t$; $-a_2 \sin t$. **5.** $a/\pi b^2$ feet per sec.

6. $bc/(a - b)$ ft. per min. **7.** $2\frac{1}{4}$ ft per sec.

8. $3/8\,\pi$ c.c. per min. **9.** $\sqrt{(2/11)}$ ft. per sec.

11. $x < -2$ and also $x > 6$.

Pages 59-60

1. 0·6064. **2.** 1·0043.

4. 0·02 lbs. per sq. in. **5.** 294 ft.; 0·79 per cent.

6. (1) $1\cdot85 \times 8\pi$ sq. inches; (2) $0\cdot4 \times (18\cdot5)^2\,\pi$ cubic inches.

7. $0\cdot7$ per cent. 8. 1 per cent.

PAGES 62-64

1. $a\cos t - 2b\sin 2t$; $-a\sin t - 4b\cos 2t$.

2. $2\pi c^2 t$ sq. ft. per sec. 3. $120\pi\, ar^3/(60r + at)^2$.

4. $-(v-b)\,v^3/\{pv^3 - av + 2ab\}$.

5. $2a$; $3a$. 6. decrease by $5'\ 46''$.

10. (i) $-\frac{1}{2}, -\frac{1}{2}$, 3 and -2, (ii) 1, 1, -3 and -2,
(iii) $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ and 1.

11. $4(3a-1)(3b-a^2) = (a-9b)^2$. 12. $3\cdot91$.

14. $3e^{-t/2}(6\cos 3t - \sin 3t)$. 15. $(5\,\pi/36)\cot 63°$.

PAGES 68-69

1. $e^{5x}(25x^4 + 40x^3 + 12x^2)$;
$2\,e^{\sin x^2}\{(\cos x^2)(1 + 2x^2\cos x^2) - 2x^2\sin x^2\}$;
$-\sin(\cos x)\sin^2 x - \cos(\cos x)\cos x$;
$6x\tan^{-1}x^2 + (6x^7 + 14x^3)/(1+x^4)^2$;
$25e^{5x}\sec^2(e^{5x})\{1 + 2e^{5x}\tan(e^{5x})\}$.

4. $a^n e^{ax+b}$.

5. $a^n(p/q)(p/q-1)(p/q-2)\ldots(p/q-n+1)(ax+b)^{p/q-n}$.

6. $(-1)^{n-1}(n-1)!\,\{a^n(ax+b)^{-n} + c^n(cx+d)^{-n}\}$.

7. $\frac{1}{2}\{\cos(x+\frac{1}{2}n\pi) - 5^n\cos(5x+\frac{1}{2}n\pi)\}$.

8. $\frac{1}{4}\{2^n\cos(2x+\frac{1}{2}n\pi) + 4^n\cos(4x+\frac{1}{2}n\pi) + 6^n\cos(6x+\frac{1}{2}n\pi)\}$.

9. $\frac{1}{8}\{4^n\cos(4x+\frac{1}{2}n\pi) + 2^{n+2}\cos(2x+\frac{1}{2}n\pi)\}$.

10. $\frac{1}{16}\{2\sin(x+\frac{1}{2}n\pi) + 3^n\sin(3x+\frac{1}{2}n\pi) - 5^n\sin(5x+\frac{1}{2}n\pi)\}$.

11. $\frac{1}{4}(a^2+9b^2)^{n/2}e^{ax}\cos\{3bx + n\tan^{-1}(3b/a)\}$
$+ \frac{3}{4}(a^2+b^2)^{n/2}e^{ax}\cos\{bx + n\tan^{-1}(b/a)\}$.

12. $\frac{1}{2}r^n e^{ax}\sin\{(b+c)x + n\phi\} + \frac{1}{2}r'^n e^{ax}\sin\{(b-c)x + n\psi\}$,
where $r^2 = a^2 + (b+c)^2$, $\phi = \tan^{-1}\{(b+c)/a\}$,
$r'^2 = a^2 + (b-c)^2$, $\psi = \tan^{-1}\{(b-c)/a\}$.

13. $(n!/2a)\{(a-x)^{-n-1} + (-1)^n(a+x)^{-n-1}\}$.

14. $(-1)^n(n!)\{16(x-2)^{-n-1} - (x-1)^{-n-1}\}$, when $n > 2$.

15. $(n!)\{3^{n+1}(1-3x)^{-n-1} - 2^{n+1}(1-2x)^{-n-1}\}$.

16. $(-1)^n n!\,\{a(a-b)^{-1}(a-c)^{-1}(x-a)^{-n-1} + b(b-c)^{-1}$
$\times (b-a)^{-1}(x-b)^{-n-1} + c(c-a)^{-1}(c-b)^{-1}(x-c)^{-n-1}\}$.

17. $(-1)^n (n!) c^{n-1} (bc - ad)(cx + d)^{-n-1}$.

18. $\frac{1}{4}(-1)^n n! \{(x + 1)^{-n-1} - (x - 1)^{-n-1}$
$- 2 \sin^{n-1}\phi \sin (n - 1) \phi/n(n - 1)\}$, where $x = \cot \phi$.

19. $(-1)^{n-1} (n - 1)! \sin^n \phi \sin n\phi$, where $\phi = \tan^{-1} (1/x)$.

20. $r^n e^{ax} \sin (bx + n\phi)$, where $r^2 = a^2 + b^2$, and $\phi = \tan^{-1} (b/a)$;
$\frac{1}{2}(1 - b)^n \cos \{(1 - b)x + \frac{1}{2}n\pi\}$
$- \frac{1}{2}(1 - b)^n \cos \{(1 + b) x - \frac{1}{2}n\pi\}$.

Pages 70-71

1. $2^4 e^{2x} (x^2 + 4x + 3)$; $6/x$; $3^3 (3x^2 - 4) \sin 3x - 6^3 x \cos 3x$;
$4! (x + a)^{-5} \log x - 2 (25 x^3 + 23 ax^2 - 13 a^2x + 3a^3)$
$\div x^4(x + a)^4$;
$(a^2 + b^2)^2 x e^{ax} \sin \{bx + 4 \tan^{-1} (b/a)\}$
$+ 4 (a^2 + b^2)^{3/2} e^{ax} \sin \{bx + 3 \tan^{-1} (b/a)\}$.

2. $a^{n-2} e^{ax} \{a^2x^2 + 2nax + n(n - 1)\}$.

3. $e^x \{(ax + b)^3 + 3an (ax + b)^2 + 3n (n - 1) a^2 (ax + b)$
$+ n(n - 1)(n - 2) a^3\}$.

4. $x \{x^2 - 3n(n - 1)\} \cos (x + \frac{1}{2}n\pi)$
$+ n\{3x^2 - (n - 1)(n - 2)\} \sin (x + \frac{1}{2}n\pi)$.

5. $m(m - 1)(m - 2) \dots (m - n + 3) a^{n-2} (ax + b)^{m-n}$
$\times \{(m - n + 2)(m - n + 1) a^2x^2 + 2n(m - n + 2)a(ax + b)x$
$+ n(n - 1)(ax + b)^2\}$.

6. $(-1)^n . (n - 4)!\, 6 x^{-n+3}$.

7. $(-1)^{n-1} \{(n - 1)! a^n (ax + b)^{-n} \sin x$
$- {}^nC_1 (n - 2)! a^{n-1} (ax + b)^{-n+1} \cos x$
$- {}^nC_2 (n - 3)! a^{n-2} (ax + b)^{-n+2} \sin x + \dots$
$+ {}^nC_r (-1)^r (n - r - 1)! a^{n-r} (ax + b)^{-n+r} \sin (x + \frac{1}{2}r\pi)$
$+ \dots\} + \log (ax + b) \sin (x + \frac{1}{2}n\pi)$.

8. $e^x \{\log x + {}^nC_1 x^{-1} - {}^nC_2 x^{-2} + {}^nC_3\, 2! x^{-3} + \dots$
$+ (-1)^{n-1} (n - 1)! x^{-n}\}$.

9. $(-1)^{n-1} (n - 3)! \{(n - 1)(n - 2) x^2 \sin^n \phi \sin n\phi$
$- {}^nC_1\, 2x (n - 2) \sin^{n-1} \phi \sin (n - 1)\phi$
$+ 2\, {}^nC_2 \sin^{n-2} \phi \sin (n - 2) \phi\}$, where $x = \cot \phi$.

Page 72

1. 0 when n is odd, and
$(-1)^{(n-2)/2} (n - 2)^2 (n - 4)^2 (n - 6)^2 \dots 2^2 . 2$ when n is even.

2. 0 when n is even, and $(n-2)^2(n-4)^2 \dots 3^2$ when n is odd.

3. 0 when n is even, and $(-1)^{(n-1)/2}(n-1)!$ when n is odd.

Pages 72-74

3. $2(-1)^{n-1}(n-1)! \sin^n \phi \sin n\phi$, where $x = \cot \phi$.

5. $(n-1)!/x$. 13. $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - n^2 y_n = 0$.

14. $-\{(n-2)^2 + m^2\}\{(n-4)^2 + m^2\}\dots\{1^2+m^2\} me^{m\pi/2}$, n odd;
$\{(n-2)^2+m^2\}\{(n-4)^2+m^2\}\dots m^2 e^{m\pi/2}$ if n is even.

17. $\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-2^2)m^2$, n even;
$\{m^2-(n-2)^2\}\{m^2-(n-4)^2\}\dots(m^2-1^2)\,.\,m$, n odd.

Pages 76-77

1. $1 - x^2/2! + x^4/4! - x^6/6! + \dots (-1)^m x^{2m}/(2m)! + \dots$

2. $1 + mx + \{m(m-1)/2!\}x^2$
$+ \{m(m-1)(m-2)/3!\}x^3 + \dots + x^m$.

3. $x - x^2/2 + x^3/3 - \dots + (-1)^{n-1}x^n/n + \dots$

4. $1 + x + x^2/2! + x^3/3! + \dots + x^n/n! + \dots$

5. $x + 1^2 \,.\, x^3/3! + 3^2 \,.\, 1^2 \,.\, x^5/5! + 5^2 \,.\, 3^2 \,.\, 1^2 \,.\, x^7/7! + \dots$

6. $1 + x + x^2/2 - x^4/8 + \dots$ 7. $x + x^3/3 + (2/15)x^5 + \dots$

8. $x - x^3/3 + x^5/5 - x^7/7 + \dots + (-1)^{n-1}x^{2n-1}/(2n-1) + \dots$

9. $1 + x + x^2/2 - x^3/3 - 11x^4/24 - x^5/5 + \dots$

10. $e^{a\pi/2}\{1 - ax + a^2x^2/2! - a(1+a^2)x^3/3!$
$+ (2^2+a^2)a^2x^4/4! + \dots\}$

11. $1 + x^2/2! + 5x^4/4! + 61x^6/6! + \dots$

12. $1 + x + x^2 + 2x^3/3 + \dots$

13. $x + x^2/2! + 2x^3/3! + 9x^5/5! + \dots$

14. $x^2 - 8x^6/6! + \dots$ 15. $x^2 - 5x^4/6 + 32x^6/45 + \dots$

18. $\log 2 + \frac{1}{2}x + \frac{1}{4}x^2/2! - \frac{1}{8}x^4/4! + \dots$

20. $1 + ax + (a^2-b^2)x^2/2! + a(a^2-3b^2)x^3/3! + \dots$
$+ \{(a^2+b^2)^{n/2}/n!\}x^n \cos\{(n \tan^{-1}(b/a)\} + \dots$

Pages 78-79

2. $\sin^{-1} b + x(1-b^2)^{-1/2} + (x^2/2!)\,b\,(1-b^2)^{-3/2}$
$+ (x^3/3!)\{(1-b^2)^{-5/2}(1+2b^2)\} + \dots$

3. $e^{-2} + 2(x+1)e^{-2} + 2^2(x+1)^2 e^{-2}/2! + 2^3(x+1)^3 e^{-2}/3! + \dots$

4. $\tan^{-1}\frac{1}{4}\pi + (x - \frac{1}{4}\pi)/(1 + \frac{1}{16}\pi^2)$
$- \pi(x - \frac{1}{4}\pi)^2/4(1 + \frac{1}{16}\pi^2)^2 + \dots$

5. $(1/\sqrt{2})\,(1 + \theta - \theta^2/2! - \theta^3/3! + \theta^4/4! + \theta^5/5! - \dots)$

6. $45 + 53\,(x - 2) + 19\,(x - 2)^2 + 2\,(x - 2)^3$.

PAGES 84-85

1. $(1/\sqrt{2})\{1 + \theta - \theta^2/2! - \theta^3/3! + \theta^4/4! + \dots\}$.

4. $\frac{1}{2} + \frac{1}{4}x - \frac{1}{8}x^3/3! + \dots$ **5.** $1/4!$

8. $2x - x^2 - 2x^3/3 + 3x^4/2 - 5x^5/3 + 3x^6/2 + \dots$

9. $m \sin\theta - \{m(m^2 - 1^2)/3!\}\sin^3\theta$
$$+ \{m\,(m^2 - 1^2)\,(m^2 - 3^2)/5!\}\sin^5\theta + \dots$$

10. The coefficient of x^n is
$$a(a^2 + 1^2)\,(a^2 + 3^2)\dots\{a^2 + (n-2)^2\}/n!,$$
or $a^2(a^2 + 2^2)\,(a^2 + 4^2)\dots\{a^2 + (n-2)^2\}/n!$, according as n is odd or even.

11. $x + x^3/6 + \dots;\ x - x^2 + 7x^3/6 + \dots$

12. $a_{2m} = 0$,
$$a_{2m+1} = 2m(2m-2)\,(2m-4)\dots 2/(2m+1)(2m-1)\dots 3.$$

13. $x + x^2/2 - 5x^4/24 + \dots;\ y - y^2/2 + \dots$

PAGES 89-90

1. (i) $yY = 2a\,(X + x)$; (ii) $Y/y + X/x = 2$;

(iii) $y^{m-1}\,Y/b^m + x^{m-1}\,X/a^m = 1$;

(iv) $\{2y\,(x^2 + y^2) + a^2y\}\,Y + \{2x\,(x^2 + y^2) - a^2 x\}\,X$
$$= a^2\,(x^2 - y^2);$$

(v) $Y - y = \{\sinh(x/a)\}\,(X - x)$;

(vi) $Y - y = a(\cot x)\,(X - x)$.

2. (i) (a) where $ax + by = 0$ intersects the curve; (b) where $by + bx = 0$ intersects the curve;

(ii) (a) where $4ax - y = 0$ intersects the curve; (b) nowhere ($x = 0$ is an asymptote);

(iii) (a) at $x = b^3$, $y = a + 3\log b + 1$; (b) nowhere.

3. (i) $2Y - 3tX - t^3 + 3at = 2b$;

(ii) $Y = (X - at)\tan\frac{1}{2}t$; (iii) $X/a\sin t + Y/b\cos t = 1$.

10. (i) $\frac{1}{2}\pi$, and $\tan^{-1}\frac{1}{2}$; (ii) $\frac{1}{2}\pi$ and $\tan^{-1}(3/5)$;

(iii) 0, i.e., the curves touch each other.

PAGE 91

1. (i) $2aY + yX = xy + 2ay$; (ii) $xX - yY = x^2 - y^2$;

(iii) $y^{m-1}\,X/b^m - x^{m-1}\,Y/a^m = xy^{m-1}/b^m - yx^{m-1}/a^m$;

(iv) $(2x^2 + 2y^2 + a^2)yX - (2x^2 + 2y^2 - a^2)xY = a^2xy$:

(v) $X - x + (Y - y) \sinh (x/a) = 0$;

(vi) $X \sin x + aY \cos x = x \sin x + ay \cos x$.

2. $20y - x + 7 = 0$; $y + 20x - 140 = 0$.

3. (i) $2x + 3ty = 3t^4 + 2t^2 - 3bt - 2a^2$;

(ii) $x + y \tan \frac{1}{2}t = a(t + \sin t + 2 \sin^2 \frac{1}{2}t \tan \frac{1}{2}t)$;

(iii) $ax \sin t - by \cos t = a^2 \sin^4 t - b^2 \cos^4 t$.

4. $yy' = 2a(x + x')$; $xy' + 2ay = (x' + 2a)y'$; $2(y'^2 + 4a^2)^{3/2}/y'^2$.

PAGE 93

4. $a \cosh^2 (x/a)$, $\frac{1}{2}a \sinh (2x/a)$.

6. $a \sin t$; $2a \sin^3 \frac{1}{2}t \sec \frac{1}{2}t$; $2a \sin \frac{1}{2}t \tan \frac{1}{2}t$; $2a \sin \frac{1}{2}t$; at; $-at \tan \frac{1}{2}t$.

7. $\frac{1}{2}$.

PAGES 98-99

1. $\tan^{-1} \{(1 + e \cos \theta)/e \sin \theta\}$. 2. $\frac{1}{4}\pi$. 3. $\frac{1}{2}\pi$.

5. $\tan^{-1} \cot (n\theta + \alpha)$. 6. $\cos^{-1} (a/r)$. 8. $l/c \sin \theta$.

10. $a^2/p^2 = 1/\theta^2 - 2/\theta^3 + 2/\theta^4 - 4/\theta^5 + 4/\theta^6$.

PAGE 104

1. (i) $\sqrt{(4a^2x^2 + 4abx + b^2 + 1)}$. (ii) $\sec x$. (iii) $\cosh (x/a)$.

4. (i) $\sqrt{(1 + 4t^2)}$; (ii) $(a^2 \sin^2 t + b^2)^{1/2} \sec^2 t$;

(iii) $2 \cos t \,.\, (1 + 4 \sin^2 t)^{1/2}$.

5. (i) $\sqrt{(r^2 + 9 \cot^2 3\theta)}$; (ii) $\frac{1}{2}(1 + 4 \tan^2 \theta) \sec^2\theta$.

PAGES 104-107

2. $(0, 0)$; $(\sqrt{3}, -\sqrt{3}/2)$; $(-\sqrt{3}, \sqrt{3}/2)$. 3. $\frac{1}{4}\pi$.

4. $y = x + a(2 + \sqrt{2})$. 7. $4a/9$. 13. $\frac{1}{2}\pi$.

14. $\frac{1}{2}\pi$. 19. $\tan \frac{1}{2}t$. 23. $y \cos^3 \theta - x \sin^3 \theta = \frac{3}{4} \sin 4\theta$.

PAGES 108-112

4. (i) continuous everywhere; (ii) discontinuous at $x = -1$;
(iii) discontinuous at $x = 0$, unless $a = 1$.

14. $6{\cdot}954$. 15. $1{\cdot}732$. 16. $1\frac{4}{7}$ miles per hour.

17. $6\frac{2}{3}$ ft. per sec. 21. $1{\cdot}38$; $4{\cdot}8$.

PAGE 117

1. $2y + x = 1,\ y = x,\ y + x + 1 = 0.$
2. $2y + x = 1,\ y = x + 1,\ y + x = 0.$
3. $y = x,\ y + 2x + 1 = 0,\ 2y + x + 1 = 0.$
4. $y = x,\ y = 2x + 3.$
5. $y = -x,\ y = x + \frac{2}{3},\ y = 4x + \frac{1}{3}.$

PAGE 119

1. $y = x,\ y = -x - 1,\ y + x = 1.$
2. $y = x,\ y = -x,\ y = -x - 1.$
3. $y = x,\ y = -2x,\ y = -2x - 1.$
4. $y = x,\ y = -\frac{1}{2}x + \frac{1}{2},\ 2y = -x - 1.$
5. $y = x,\ y = \frac{1}{2}x,\ 2y = x + 1.$

PAGE 121

1. $3x + 1 = 0.$
2. $y = \pm 1.$
3. $y = \pm 1,\ x = \pm 1.$
4. $y = 0.$
5. $y = b,\ x = -a.$

PAGES 124-125

1. $y = x - \frac{1}{3}a.$
2. $y = x,\ y = x + 1,\ y = -x.$
3. $x = \pm a,\ y = 0.$
4. $y = x,\ y = 0,\ y = x \pm 1.$
5. $\pm y = x + \frac{1}{2}b,\ x = b.$
6. $\pm y = x + a,\ x = 0.$
7. $y = 0,\ x = 0,\ y = 3x/2 + 3.$
8. (i) $y = \pm x.$ (ii) $x = 0.$
9. $x = \pm a,\ y = \pm x.$
10. $x = \pm a,\ y = \pm b.$
11. $y = x,\ y = 2x,\ y = -x - 1,\ y = -2x - 1.$
12. $y = x,\ y = 2x + 1,\ y = -x - 2.$
13. $y = x,\ y = -x,\ y = -\frac{1}{2}x - \frac{1}{2}.$
14. $x = 0,\ y = 0,\ y = 2x + \frac{3}{2},\ 2y + 4x = 15.$
15. $x = 2,\ x = 1,\ y = -x + 2.$
16. (i) $\alpha_1 x + \beta_1 y + \gamma_1 = 0,\ \alpha_2 x + \beta_2 y + \gamma_2 = 0;$
    (ii) $y = x - 1/3;$
    (iii) $x = \pm a,\ y = x + a,\ y = -x + a.$
17. $x = 2a,\ y = x + a,\ y = -x - a.$
18. $y = -x + 2a/3.$
19. $x = 3,\ y = x + 1,\ y = x + 2.$ In the first quadrant one branch of the curve is above $y = x + 2$ and another below $y = x + 1$. In the third quadrant the curve lies between the asymptotes. In the first quadrant the curve lies to the right of $x = 3$ and in the fourth quadrant to the left of $x = 3$.

21. Above the asymptote.

22. For positive values of x, the curve lies above or below the asymptote according as $a_0^3 b_2 - a_0^2 a_1 b_1 + a_0 a_1^2 b_0 - a_0^2 a_2 b_0$ is positive or negative. The reverse is the case for negative values of x.

Pages 126-127

2. $b^2(x+a) = a^2(y+b)$.

3. The asymptotes are $x = \pm a$, $y = x + a$, $y = \pm x - \frac{1}{2}a$; the quartic curve is $bxy(x^2 - y^2) + a(b^2 - a^2)(x^2 + y^2 - a^2) = 0$.

Page 127

2. $y = 0$; $y^2 = ax$.

Page 128

1. $y = 0$. **2.** $y = 0$.

3. $x = \pm \frac{1}{2}\pi, \pm \frac{3}{2}\pi, \ldots$ **4.** $x = 0, \pm \pi, \pm 2\pi, \ldots$

Page 130

1. $r \sin(\theta \pm \frac{1}{6}\pi) = 2/\sqrt{3}$. **2.** $r \sin\theta = a$. **3.** $r\cos\theta = 8$.

4. $r\cos\theta = 4$. **5.** $r\sin\theta = 2k\pi$; $k = \pm 1, \pm 2, \ldots$

6. $r\sin\theta = 2$. **7.** $r\sin\theta = \pi$. **8.** $r\cos\theta = \pm a$.

9. $r\sin\theta = \pm \frac{1}{2}a$, $r\cos\theta = \pm \frac{1}{2}a$.

10. $r\sin\theta = a$. **11.** $r = 1$. **12.** $r = 3$.

Pages 130-131

1. $y = 0$, $y = 1$, $x = 0$, $x = 1$.

2. $x + 1 = 0$, $y = 0$, $x + y = 0$.

3. $x + 2y + 1 = 0$, $x + 2y = \pm 2\sqrt{2}$.

4. $x = 0$; $y = 0$; $2y = 3x + 6$. **5.** $x - y = \pm a$; $x = \pm a$.

6. $y = x$. **7.** $x = a$; $x = -a$.

8. $y = x + 4$, $2y = x - 2 \pm 3\sqrt{3}$.

9. $y = x$; $y = 2x + 2$; $y = 2x + 1$.

10. $r\sin(\theta - \frac{1}{4}\pi) = \pm 2$; $r\cos(\theta - \frac{1}{4}\pi) = \pm 2$.

11. $n\theta = m\pi$, where m is an integer, if $|n| > 1$; no asymptote if $|n| < 1$

12. $r\sin\theta = a$. **13.** $r\sin\theta = 2a$, $r = a$. **14.** $x = a$, $x = -a$.

17. $y - x + 7/6 = 0, y - 3x + 3/2 = 0, 2y + x + 5/3 = 0$;
$106y - 381x + 105 = 0$.

19. $x + y = \pm 2\sqrt{2}$; $x + 2y + 2 = 0$.

20. $x = 3$; $x = 2$; $y = 3$; $y = -3$.

Above $y = 3$ in the first quadrant, below $y = 3$ in the 2nd quad., above $y = -3$ in the 3rd quad., and below $y = -3$ in the 4th quad. To the left of $x = 2$ and to the right of $x = 3$.

PAGE 136

1. $(4a/3) \sin(\psi/3)$. **2.** $4a \cos \psi$. **3.** $c \tan \psi$.

4. $\sqrt{8}$. **5.** $1/\sqrt{2}$. **6.** $(2/\sqrt{a})(x + a)^{3/2}$.

7. $(1/6a)(4a + 9x)^{3/2} x^{1/2}$. **8.** $a \sec(x/a)$. **9.** $3(axy)^{1/3}$.

PAGES 138-139

1. $2\sqrt{(2ar)}/3$. **2.** r^3/a^2.

3. $a^2/3r$. **4.** $a^n r^{-n+1}/(n + 1)$.

5. $(r^2 + a^2)^{3/2}/(r^2 + 2a^2)$. **6.** a^2b^2/p^3.

8. $r(a^2 + r^2)^{3/2} a^{-3}$. **9.** $\frac{1}{2} a$.

10. $\sqrt{(8r^3)}/\sqrt{a}$. **11.** $a^n r^{-n+1}/(n + 1)$.

12. $a^{1/2}(3a - 2r)^{3/2}/3(2a - r)$. **13.** $\sqrt{(r^2 - a^2)}$.

PAGES 141-142

1. $(c^2 + s^2)/c$. **2.** $\sqrt{(16a^2 - s^2)}$.

3. $1/36$. **4.** $37\sqrt{(37)}/10$.

5. $85\sqrt{(17)}/2$; $5\sqrt{2}$. **7.** $(b^4 \cos^2\theta + a^4 \sin^2\theta)^{3/2}/u^3a^4b^4$.

10. $2\sqrt{2}$ for each.

PAGES 144-145

1. (i) $0, -2\frac{3}{4}$; (ii) $-36, -7\frac{1}{8}$. **8.** $4r/3$.

PAGES 149-150

2. Pts. of inflexion at $x = 3$, $(28 \pm \sqrt{3})/11$. Pt. of undulation at $x = 2$.

5. No.

PAGES 153-155

5. $a/3$. **21.** $\{x - (y/a)\sqrt{(y^2 - a^2)}, 2y\}$.

30. $(9am^2, -6am)$.

PAGES 164-165

1. (i) $2xy - y^2 = 0$; $3xy + 7x^2 = 0$. **7.** $(a, 0), (-a, 0), (0, -a)$.

10. (i) $(0, 0)$ is a conjugate point;

(ii) $(0, 3)$ is a conjugate point and at $(2, 3)$ there is a single cusp of the first species.

(iii) Node at $(3, 2)$.

12. There is a single cusp of the first species at $(1, -1)$.

PAGES 183 184

5. $x + y = a$. **6.** The origin is a node and also a point of inflexion.

8. $x = 3$; $y = x + 1$; $y = x + 2$.

9. There is a single cusp of the first species at the origin. Asymptote is $y = x + \frac{1}{3}a$.

11. Conjugate point. **14.** Node.

16. $y = 0$; $x = 0$; $y = \pm x$.

21. Tangent parallel to axes is $x = 0$; points of inflexion at $x = -(1/\sqrt{3})\sqrt{\{-5 + \sqrt{(28)}\}}$.

PAGES 187-188

1. (i) $(x^2 + 2xy - y^2)/\{(x+y)^2 + (x^2 + y^2)^2\}$;

$(y^2 + 2xy - x^2)/\{(x+y)^2 + (x^2 + y^2)\}$.

(ii) $2x/a^2$; $2y/b^2$. (iii) $x^y \cdot y/x$; $x^y \log x$.

(iv) $2x/(x^2 + y^2)$; $2y/(x^2 + y^2)$.

PAGE 191

1. (i) $-(ax + hy)/(hx + by)$;

(ii) $-(y^x \log y + yx^{y-1})/(xy^{x-1} + x^y \log x)$.

2. $1 + \log xy - x(x^2 + y)/y(x + y^2)$.

3. $2x\{\cos(x^2 + y^2)\}(1 - a^2/b^2)$.

4. If $f = x^2y$, then $\partial f/\partial x = 2xy$, $\partial^2 f/dx^2 = 2y$, $\partial f/\partial y = x^2$, $\partial^2 f/\partial x \partial y = 2x$, $\partial^3 f/\partial x^2 \partial y = 2$, and all the higher differential coefficients are zero. $df/dx = 2xy - x^2(2x + y)/(x + 2y)$.

PAGES 200-201

1. -22 cu. in. in volume and $-4{\cdot}8$ sq. in. in surface.

2. $3{\cdot}02$ cu. in., $2{\cdot}82$ sq. in. The calculated values will be too large by these amounts.

Pages 204-207

1. (i) 0; (ii) $-3/2$.

6. $\{2a^2(x^3+a^2y)(y^3+a^2x)-3x^2(y^3+a^2x)^2-3y^2(x^3+a^2y)^2\}$
$-(y^3-a^2x)^3$.

7. $\{2h(ax+hy+g)(hx+by+f)-a(hx+by+f)^2$
$-b(ax+hy+g)^2\}/(hx+by+f)^3$.

17. $-r\sin\theta$; $-(\sin\theta)/r$.

Pages 211-212

1. $x^2+y^2=a^2$. The given family of curves consists of all the straight lines which are at a distance a from the origin.

2. $y^2=4x$.
3. $x^2/a^2-y^2/b^2=1$.
4. $4x^3+27ay^2=0$.
5. $4xy=c^2$.
6. $(p-1)^{p-1}x^p+p^p ay^{p-1}=0$.
7. $r^{n/(m-n)}=a^{n/(m-n)}\cos\{n\theta/(m-n)\}$, where r, θ are the polar coordinates of (x, y).
8. $(x^2+y^2-c^2)^2=4a^2(x^2+y^2)$.
9. $y^2-4x-4=0$.
10. $(x+y+1)^2=2(x^2+y^2)$.
11. $x^6+4ay=0$.
12. $4x^3=27y$.
13. $x^{2/3}+y^{2/3}=l^{2/3}$.

Page 216

1. (i) $r\cos\theta+a\sin^2\theta=0$;
(ii) $r^2(e^2-1)-2ler\cos\theta-2l^2=0$,
(iii) $r^{3/4}=a^{3/4}\cos(3\theta/4)$, (iv) $r\cos^{n-1}\{\theta/(n-1)\}=a$.

2. (i) $r\cos^2\frac{1}{2}(\theta-\alpha)=p$; (ii) a circle through the pole;
(iii) $r^{n/(n+1)}\cos\{n\theta/(n+1)\}=a^{n/(n+1)}$;
(iv) $r^{n/(1-n)}=a^{n/(1-n)}\cos\{n\theta/(1-n)\}$.

3. $(m+n)^{m+n}x^my^n/m^mn^n=c^{m+n}$.

5. $4x^2y^2=c^4$, where πc^2 is the given area.

Pages 217-218

1. $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

Pages 220-222

2. $c^2x^2y^2=a^4y^2+b^4x^2$.

3. (i) $r^2=4(a^2\cos^2\theta+b^2\sin^2\theta)$; (ii) $r^2=4a^2\cos 2\theta$.

6. $x^{2/(2-n)} + y^{2/(2-n)} = a^{2/(2-n)}$.

8. $x^{mp/(m+p)} + y^{mp/(m+p)} = c^{mp/(m+p)}$.

10. $y + \frac{1}{2}gx^2/v_0^2 = v_0^2/2g$. 11. A circle.

12. $x \pm y = \pm k$. 16. $x = \pm a$.

17. $2p^2x - 2py + (2pq + lp^2 + l) = 0$, where $l =$ semi-latus rectum.

Pages 227-229

4. $x = 0, 1, 3$. Max. at $x = 1$, min. at $x = 3$, neither at $x = 0$.

5. Min. at $x = 0$, max. at $x = 6/5$.

19. Max $= c^2/(a + b)$ when $\tan\theta = \pm\sqrt{(a/b)}$.

20. $\pi/\sqrt{(ab - b^2)}$. 21. $27\sqrt{3}$ sq. in. 22. $1/e$.

Page 230

1. (i) Min. $= 2c\sqrt{(ab)}$. Max. $= -2c\sqrt{(ab)}$.
   (ii) Max. and min. values are respectively $1 \pm \cos\alpha$

2. Radius of semicircle = height of rectangle = $20/(4 + \pi)$ ft.

4. Depth $= \sqrt{(\frac{2}{3})}2a$; breadth $= \sqrt{(\frac{1}{3})}2a$ 6. 13 9 ft.

Pages 230-233

2. No. max. or min.

3. r being integral, $x = r\pi/(n + 1)$ gives maxima and minima alternately, beginning with $r = 1$, omitting, when n is even, those values of r which equal zero or a multiple of $n + 1$.

4. 160 and 0. 5. $ap/(p + q)$, $aq/(p + q)$.

6. $\frac{1}{2}$. 10. $25\sqrt{2}$ ft.

11. $3\sqrt{3}\,ab/4$. 18. $3\sqrt{3}/4$ ft.

20. Eccentric angle of $P = \frac{1}{4}\pi$.

22. 164 sq in nearly; $\sin^{-1}(\sqrt{2} - 1)$.

24. $a - b$. 25. Max. value $4/e$; min. value 0.

26. Length 2 ft., girth 4 ft.; yes, length should now be $1\frac{3}{4}$ ft., girth $4\frac{1}{4}$ ft.

27. 38 99. 29. $3\frac{5}{9}$.

Pages 236-237

1. 1. 2. $\frac{1}{2}$. 3. $-\frac{1}{3}$. 4. 1.

5. $\frac{3}{2}$. 6. $\log(a/b)$. 7. $\frac{1}{3}$. 8. 0.

9. -1. 10. $\frac{1}{2}$. 11. 4. 12. 2.
13. $81/20$. 14. ∞. 15. $1/18$.

Page 241

1. 1. 2. $\frac{1}{2}$. 3. ∞. 4. 0.
5. 0. 6. 3. 7. $-\frac{1}{2}e$. 8. -2.
9. 0. 10. 0. 11. 0. 12. a.
3. $-\frac{1}{2}$. 14. -1. 15. $-\frac{1}{8}$. 16. $-\infty$.
17. $\log a$. 18. 1. 19. 1. 20. $e^{1/3}$.
21. ∞. 22. 1. 23. 0. 24. $1/e$.
25. $e^{1/12}$. 26. c on the right, $-c$ on the left.
27. 1. 28. $1/e$. 29. 1. 30. $e^{-1/2}$.

Pages 244-245

1. $1/\sqrt{2}$. 2. No; yes, ∞ and 0 respectively.
6. $e^{1/6}$. 7. $-1/3$. 8. (i) $\frac{1}{2}$; (ii) $a = -\frac{5}{2}, b = -\frac{3}{2}$.
9. (i) -15; (ii) $8/69$ 10. $\frac{1}{3}$. 11. 3.
12. (i) 1; (ii) -1; (iii) -1. 13. 0; 0. 14. 0·995.
15. $\{\log (a/b)\}/\{\log (c/d)\}$; -1.

Pages 250-251

2. $1 + ax + (a^2 - b^2)\, x^2/2! + a(a^2 - 3b^2)\, x^3/3! + \ldots$
Remainder after n terms is
$$(a^2 + b^2)^{n/2}\, (n!)\, x^n e^{a\theta x} \cos\{b\theta x + n \tan^{-1} (b/a)\}.$$
3. $(2n + 1)\pi/2 - x + x^3/3 - x^5/5 + x^7/7 - \ldots$
4. $x = b$, 7th; $x = c$, 8th; at $x = d$ the function's definition itself fails.
6. $1 - \sqrt{(7/12)}$. 7. $x = \pm 1$.

Pages 252-256

1. $x + y = 0$, $x + 2y + 1 = 0$.
2. $6y - 6x + 7 = 0$, $2y - 6x + 3 = 0$, $6y + 3x + 5 = 0$.
3. $x = 2y - 14a$, $x = 3y + 13a$, $x - y = a$, $x - y = 2a$.
7. $(\frac{9}{8}, 3)$, $(\frac{1}{8}, -3)$. 8. 1. 9. convex.
1. $(a + b)\,(x^2 + y^2) = 2x + 2y$.
$\varrho = y^2/c$. 15. $y = a$, $x = 0$.

20. $\{\sin(C-A)+\sin B(\cos A-\cos C)\}/\{\sin(B-C)$
$+\sin A(\cos C-\cos B)\}$

23. $r^2 = 4a^2\cos 2\theta$.

24. A straight line parallel to the fixed straight line.

27. Max. at $x = 1$, min. at $x = 6$.

28. Max. at $x = 2$, min. at $x = 2\frac{1}{11}$.

29. Max. at $x = 1$, min. at $x = 2$.

32. $1/e$. 33. $(16\sqrt{3})\pi/27$ cu. ft.

34. $2\pi\{1-\sqrt{(2/3)}\}$ radian.

40. $(-\frac{1}{2}\log 2, 1/\sqrt{2})$.

42. (i) 4; (ii) 0 if $a<1$, b if $a>1$. 43. ae.

44. (i) $-\frac{1}{2}$; (ii) $1/e$; (iii) $2k$.

45. 1. 46. (i) 4, (ii) ∞. 47. $\frac{2}{3}$.

48. (i) $(\log a-1)/(\log a+1)$; (ii) $\sqrt{(ab)}$.

50. $x+x^2+x^3/3+\ldots$; $2^{n/2}\,.\,\sin\frac{1}{4}n\pi\,.\,x^n/n!$.